INTEGRATION OF OSCILLATORY AND SUBANALYTIC FUNCTIONS

RAF CLUCKERS, GEORGES COMTE, DANIEL J. MILLER,
JEAN-PHILIPPE ROLIN, and TAMARA SERVI

Abstract

We prove the stability under integration and under Fourier transform of a concrete class of functions containing all globally subanalytic functions and their complex exponentials. This article extends the investigation started by Lion and Rolin and Cluckers and Miller to an enriched framework including oscillatory functions. It provides a new example of fruitful interaction between analysis and singularity theory.

Contents

1. Introduction

In this article we prove the stability under parameterized integration of a class of functions containing all globally subanalytic functions and their complex exponentials, with methods pertaining to subanalytic geometry. Note that the theories of holonomic D-modules and holonomic distributions (and, further away, of ℓ-adic cohomology and of motivic integration) have the richness of combining geometry with Fourier transforms, and these theories all have found far-reaching applications. Applications

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of our setting are to be expected, but are not the content of the present article. Let us just mention that, in the context of motivic and p-adic integration (see [7]), similar stability results have found recent applications in the Langlands program (see [5], [6]). The stability under integration of certain classes of real functions has already been considered in [8], [10], [21], and [16], but none of these classes allows oscillatory behavior, let alone stability under Fourier transforms. Let us explain our results in detail.

Definition 1.1
A set $X \subseteq \mathbb{R}^m$ is *globally subanalytic* if, in any standard Euclidean chart $\mathbb{R}^m$ of $\mathbb{P}^m(\mathbb{R})$, the image of X in $\mathbb{P}^m(\mathbb{R})$ is a subanalytic subset of this chart in the sense of [3] and [11]. Equivalently, $X \subseteq \mathbb{R}^m$ is globally subanalytic if it is the image under the canonical projection from $\mathbb{R}^{m+n}$ to $\mathbb{R}^m$ of a globally semianalytic subset of $\mathbb{R}^{m+n}$ (i.e., a set $Y \subseteq \mathbb{R}^{m+n}$ such that, in a neighborhood of every point of $\mathbb{P}^1(\mathbb{R})^{m+n}$, Y is described by a finite number of analytic equations and inequalities). Given a set $X \subseteq \mathbb{R}^m$, a map $f : X \to \mathbb{R}^n$ is globally subanalytic if its graph is a globally subanalytic subset of $\mathbb{R}^{m+n}$. (This definition implies that X is a globally subanalytic subset of $\mathbb{R}^m$, since the collection of globally subanalytic sets is closed under projections.)

In model-theoretic terms, a set is globally subanalytic if and only if it is definable in the structure $\mathbb{R}_{\text{an}}$, the expansion of the ordered real field by all restricted analytic functions (as defined in [31]). By [12], [28], and [29], this is an o-minimal structure, and therefore, the reader may refer, for instance, to [30] and [32] for the basic geometric properties of globally subanalytic sets and functions that we will use in the rest of this article.

For the sake of brevity, from now on we will use the word “subanalytic” as an abbreviation for the phrase “globally subanalytic.” So in this usage of the word, the natural logarithm $\log\colon (0, +\infty) \to \mathbb{R}$ and the trigonometric functions $\sin\colon \mathbb{R} \to \mathbb{R}$ and $\cos\colon \mathbb{R} \to \mathbb{R}$ are not subanalytic, although the restriction of any one of these functions to any compact subinterval of its domain is subanalytic.

Given a subanalytic set $X \subseteq \mathbb{R}^m$, we denote by $\mathcal{S}(X)$ the algebra of all real-valued subanalytic functions on X, and we write

$$\mathcal{S} := \{\mathcal{S}(X) : m \in \mathbb{N}, X \subseteq \mathbb{R}^m \text{ subanalytic}\}$$

for the system of all real-valued subanalytic functions.

Our aim is to provide a full description of the smallest system

$$\mathcal{E} := \{\mathcal{E}(X) : m \in \mathbb{N}, X \subseteq \mathbb{R}^m \text{ subanalytic}\}$$

such that $\mathcal{E}(X)$ is a $\mathbb{C}$-algebra of complex-valued functions on $X \subseteq \mathbb{R}^m$ satisfying

$$\mathcal{S}(X) \cup \left\{ \mathrm{e}^{\mathrm{i}f} : f \in \mathcal{S}(X) \right\} \subseteq \mathcal{E}(X) \tag{1}$$

and such that $\mathcal{E}$ is stable under integration.

Here stability under integration for $\mathcal{E}$ means that if $X \subseteq \mathbb{R}^m$ is a subanalytic set, $n \in \mathbb{N}$, and $f \in \mathcal{E}(X \times \mathbb{R}^n)$ is such that $f(x, \cdot) \in L^1(\mathbb{R}^n)$ for all $x \in X$, then the function $F\colon X \to \mathbb{C}$ defined by

$$F(x) = \int_{y \in \mathbb{R}^n} f(x, y)\, \mathrm{d}y \quad \text{for } x \in X, \tag{2}$$

is in $\mathcal{E}(X)$.

Note that the existence of $\mathcal{E}$ is guaranteed by the fact that the collection on the left-hand side of (1) is contained in the class of all complex-valued measurable functions, a class stable under parameterized integration. We will describe in detail the system $\mathcal{E}$ in the next section. Our main result is that $\mathcal{E}$ coincides with the system $\mathcal{C}^{\exp}$ of $\mathbb{C}$-algebras $\mathcal{C}^{\exp}(X)$ defined in Definition 2.7 (see Remark 2.14(1)), for which we have an explicit description of the generators (Definition 2.15). It is worth noting that the generators of the algebra $\mathcal{E}(X)$ are defined in terms of one-variable integrals of a particularly simple form (see Definition 2.5).

A strong motivation to allow oscillatory functions in our system comes from singularity theory, where oscillatory integrals have been heavily investigated for decades (for an introduction, and among numerous other references, see in particular [2], [23], [33]). A series of preparation and monomialization results (see [8], [9], [20], [24], [25]) for subanalytic functions and their logarithms provides a powerful tool for studying the nature of oscillatory integrals with subanalytic phase and amplitude.

As indicated in [21, Introduction], the idea of using a preparation theorem to understand the integration of subanalytic functions was suggested by L. van den Dries and, indeed, was successfully used in [21] and [10], where it was proved (using [21, Théorème 1] and [10, Proposition 1], or directly as [10, Theorem 1′]) that the parameterized integrals of subanalytic functions belong to the class $\mathcal{C} := (\mathcal{C}(X))_X$ of constructible functions. (The algebra $\mathcal{C}(X)$ of functions on the subanalytic set X is generated as a $\mathbb{C}$-algebra by the subanalytic functions on X and their logarithm (see Definition 2.1).) In particular, the function volume of fibers of a subanalytic family and the density function along a subanalytic set also belong to the class $\mathcal{C}$ (see [10]).

The question of finding a system of $\mathbb{C}$-algebras of functions that contains $\mathcal{S}$ and that is stable under parameterized integration has been attacked and solved in [8] (see also [9]), where the authors showed that the class $\mathcal{C}$ itself is stable under parameterized integration (see [16] for an interesting subcollection of $\mathcal{C}$, also stable under integration). Here again the main tool of proof is a preparation theorem for functions of $\mathcal{C}$. Note that the class $\mathcal{C}$ is a class of functions definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$, the expansion of $\mathbb{R}_{\mathrm{an}}$ by the full real exponential function.

As already mentioned, the problem we address and solve here is the problem of explicitly describing a system of $\mathbb{C}$-algebras (actually the smallest), stable under parameterized integration, containing $\mathcal{S}$ and containing the complex-valued oscillatory functions $\mathrm{e}^{\mathrm{i}f}$, for all subanalytic functions f. Since we consider oscillatory functions, we are no longer in an o-minimal setting. However, the preparation results mentioned above (see Section 3) prove extremely useful and powerful even for dealing with oscillatory functions. To prove our results, we combine these preparation techniques with the theory of continuously uniformly distributed maps (see Section 6), a new ingredient in this context.

Oscillatory integrals are central in many branches of mathematics and physics. Following Stein [27], an oscillatory integral of the first kind is a parameterized integral $I(x)$, $x \in \mathbb{R}$, defined by

$$I(x) = \int_{y \in \mathbb{R}^n} f(y)\mathrm{e}^{\mathrm{i}x\Phi(y)}\,\mathrm{d}y, \tag{3}$$

where the *amplitude* f and the *phase* Φ are in general C^∞-functions. The principle of stationary phase asserts, when the phase Φ has no critical point on the support of f (assume for simplicity that f has compact support), that $x \mapsto I(x)$ is in $\mathscr{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing functions. As a consequence, the asymptotic behavior of $I(x)$ at $+\infty$, modulo $\mathscr{S}(\mathbb{R})$, presents some interest only at critical points of the phase. If the phase is analytic, then one can show that this asymptotic behavior only depends on the Taylor series of the amplitude function at critical points of the phase and that $I(x)$ can be expanded in an asymptotic series

$$\sum_p x^{-p/r} \sum_{k=0}^{n-1} c_{p,k} \log^k(x),$$

where r is a positive integer not depending on f and p is an element of $\mathbb{N} \setminus \{0\}$ (see [23, Section 7], [2, Chapter 7]). Using Hironaka's resolution of singularities on the phase function, one can prove this result by reducing to the case of a monomial phase. The exponents $-p/r$ and k are related to the monodromy of the phase, in case the phase has an isolated singular point in the complex domain: $\mathrm{e}^{2\pi\mathrm{i}(\frac{p}{r}-1)}$ is actually an eigenvalue of multiplicity at least $k+1$ of the monodromy operator of the phase (see [23] for more details). Furthermore, the principal part of the exponents $-p/r$, called the *oscillation index* (see [2, Section 6.1.9]), can be computed in terms of Newton's diagram of the Taylor expansion of the phase at its critical point (see [2], [33]).

Similarly, in this article, we estimate and compare the asymptotics at infinity of different terms appearing in our parameterized integrals, namely, integrals as in (2), and in this situation the preparation theorem for constructible functions (Proposition 3.10) appears as the counterpart of Hironaka's theorem. Of course, in our general context, no geometric interpretation for exponents appearing in the asymptotics considered can be given, but there might be connections with the classical cases still to be discovered.

An oscillatory integral of the second kind has the form

$$I(x) = \int_{y\in\mathbb{R}^n} f(x,y)\mathrm{e}^{\mathrm{i}\Phi(x,y)}\,\mathrm{d}y, \tag{4}$$

where now $x = (x_1, \dots, x_m)$ is a tuple of variables. A classical example of an oscillatory integral of the second kind is given by Fourier transforms. A second more complicated example is given by the Fourier integral operator (see [14], [27]), which plays a role in approximating the solutions of a large class of PDEs (e.g., the wave equation). A natural question arises: how does one describe the nature of (4) according to the nature of the amplitude and of the phase?

Note that in (4) the parameters x are "intertwined" with the integration variables y in the expressions for the amplitude f and the phase Φ. If we consider oscillatory integrals of the second kind with subanalytic amplitude and phase, then the aforementioned preparation results prove a very powerful tool for monomializing the phase while respecting the different nature of the variables x and y.

The main result of this article (Theorem 2.12) implies that oscillatory integrals (of the first and second kinds) with subanalytic phase and amplitude belong to the system $\mathscr{E}$. Moreover, still by the stability of $\mathscr{E}$ under integration, oscillatory integrals with subanalytic phase and amplitude in $\mathscr{E}$ still belong to $\mathscr{E}$.

In particular, for $X \subseteq \mathbb{R}^m$ subanalytic, the algebra $\mathscr{E}(X\times\mathbb{R})$ is stable under taking parametric Fourier transforms:

$$\begin{aligned}&\text{if } f(x,t)\in\mathscr{E}(X\times\mathbb{R}) \text{ and } \forall x\in X, f(x,\cdot)\in L^1(\mathbb{R}),\\ &\text{then } \hat{f}(x,y) = \int_{\mathbb{R}} f(x,t)\mathrm{e}^{-2\pi\mathrm{i}yt}\,\mathrm{d}t \in \mathscr{E}(X\times\mathbb{R}).\end{aligned} \tag{5}$$

On the other hand, $\mathscr{E}$ can also be viewed as the smallest system of $\mathbb{C}$-algebras that contains the class $\mathscr{C}$ of constructible functions and is stable under composition with subanalytic functions and parametric Fourier transforms (see Remark 2.14(3)). Since there are not many systems (of algebras) of functions which are stable under Fourier transforms, we would like to insist in this Introduction on the fact that $\mathscr{E}$ is such a system, which is moreover fully described by its generators.

Like $\mathscr{E}(\mathbb{R}^n)$, the space of Schwartz functions $\mathscr{S}(\mathbb{R}^n)$ is also an algebra that is stable under taking Fourier transforms. Since the Fourier transform operator

$$\mathscr{F}\colon \big(\mathscr{S}(\mathbb{R}^n), \|\cdot\|_2\big) \to \big(\mathscr{S}(\mathbb{R}^n), \|\cdot\|_2\big)$$

is continuous, using the density of $\mathscr{S}(\mathbb{R}^2)$ in the space $L^2(\mathbb{R}^n)$, one can extend $\mathscr{F}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ to

$$\widetilde{\mathscr{F}}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \tag{6}$$

One thus obtains the classical stability of $L^2(\mathbb{R}^n)$ under the Fourier–Plancherel extension $\widetilde{\mathscr{F}}$ of the Fourier transform $\mathscr{F}$. In Section 7 we prove that $\mathcal{E}$ is even stable under the extension $\widetilde{\mathscr{F}}$ of the Fourier transform: the image of $\mathcal{E}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ under $\widetilde{\mathscr{F}}$ is $\mathcal{E}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ (see Theorem 8.3). To this end, we need to develop in Section 8 elements of a theory of uniformly distributed families of maps.

Let us also mention that, since the function $\mathrm{e}^{-|x|}$ is in $\mathcal{E}(\mathbb{R})$ (see Example 7.4), one may interpolate *families of exponential periods* with functions from $\mathcal{E}$. More precisely, by following [4] and [17, Section 4.3], a real number a is called an *exponential period* if there exist $\Delta \subset \mathbb{R}^n$ (for some $n \in \mathbb{N}$) and functions $f, g\colon \Delta \to \mathbb{R}$ such that Δ, f, and g are semialgebraic over $\mathbb{Q}$ (i.e., they are described by first-order formulas in the language of ordered rings with no other constant symbols than rational numbers) and

$$a = \int_{y \in \Delta} f(y)\mathrm{e}^{g(y)}\,\mathrm{d}y.$$

A natural version in families of this concept is the following. Let $X \subseteq \mathbb{R}^m$, let $\Delta \subseteq X \times \mathbb{R}^n$, and let $f, g\colon \Delta \to \mathbb{R}$ be semialgebraic over $\mathbb{Q}$. Suppose that, for each $x \in X$,

$$a(x) = \int_{y \in \Delta_x} f_x(y)\mathrm{e}^{g_x(y)}\,\mathrm{d}y$$

is finite, where $\Delta_x = \{y \in \mathbb{R}^n : (x, y) \in \Delta\}$, $f_x(y) = f(x, y)$, and $g_x(y) = g(x, y)$. Then the collection $\{a(x) : x \in X \cap \mathbb{Q}^m\}$ forms a natural family of exponential periods. Suppose that there is a constant N such that $g < N$ on Δ. It then follows from the stability under the integration of $\mathcal{E}$ (see Theorem 2.12) and Example 7.4 that the interpolating function $\mathbb{R} \ni x \mapsto a(x) \in \mathbb{R}$ belongs to $\mathcal{E}(X)$.

Finally, the work in this article can be seen as addressing a question raised by D. Kazhdan at the 2009 Model Theory Conference, about a possible model-theoretic understanding of real oscillatory integrals, an analogy to the understanding of motivic oscillatory integrals in [7] and [15].

2. Notation, main results, and layout of this article

This section states the main definitions, theorems, and corollaries of the article. We proceed to construct $\mathcal{E}$ by first defining some systems of rings of functions intermediary between $\mathcal{S}$ and $\mathcal{E}$.

Definition 2.1

For each subanalytic set $X \subseteq \mathbb{R}^m$, define $\mathcal{C}(X)$ to be the ring of real-valued functions on X generated by

$$\mathcal{S}(X) \cup \{\log f(x) : f \in \mathcal{S}(X), f > 0\}.$$

We call $\mathcal{C}(X)$ the *ring of constructible functions* on X, and we say that a function is *constructible* if it has a subanalytic domain X and is a member of $\mathcal{C}(X)$. Write

$$\mathcal{C} := (\mathcal{C}(X))_{X \text{ is subanalytic}}$$

for the system of all constructible functions. Thus, $f \in \mathcal{C}(X)$ if and only if f can be expressed as a finite sum of finite products of the form

$$f(x) = \sum_j f_j(x) \prod_k \log f_{j,k}(x) \tag{7}$$

with $f_j, f_{j,k} \in \mathcal{S}(X)$ and $f_{j,k} > 0$.

It is easy to see that any constructible function can be defined as a parameterized integral of a subanalytic function, and it was shown in [8, Theorem 1.3] that the constructible functions are stable under integration. Therefore, the constructible functions form the smallest class of functions defined on the subanalytic sets that is stable under integration and that contains all subanalytic functions.

It follows that

$$\mathcal{C}(X) \cup \{\mathrm{e}^{\mathrm{i} f(x)} : f \in \mathcal{S}(X)\} \subseteq \mathcal{E}(X)$$

for each subanalytic set X. This leads us to the following definition.

Definition 2.2
For each subanalytic set $X \subseteq \mathbb{R}^m$, define $\mathcal{C}^{\exp}_{\text{naive}}(X)$ to be the ring of functions on X generated by

$$\mathcal{C}(X) \cup \{\mathrm{e}^{\mathrm{i} f(x)} : f \in \mathcal{S}(X)\}.$$

Write

$$\mathcal{C}^{\exp}_{\text{naive}} := (\mathcal{C}^{\exp}_{\text{naive}}(X))_{X \text{ is subanalytic}}.$$

Thus, $f \in \mathcal{C}^{\exp}_{\text{naive}}(X)$ if and only if f can be written as a finite sum

$$f(x) = \sum_{j=1}^{J} f_j(x) \mathrm{e}^{\mathrm{i}\phi_j(x)}, \quad \text{with } f_j \in \mathcal{C}(X) \text{ and } \phi_j \in \mathcal{S}(X).$$

The elements of $\mathcal{C}^{\exp}_{\text{naive}}(X)$ are complex-valued functions. Hence, it is convenient to give the following definition.

Definition 2.3
If $f : X \to \mathbb{C}$ is such that its real and imaginary components are in $\mathcal{S}(X)$ (resp., in $\mathcal{C}(X)$), then we call f a *complex-valued subanalytic* (resp., *constructible*) function. Note that if $\phi(x)$ is a bounded subanalytic function, then $\mathrm{e}^{\mathrm{i}\phi(x)}$ is a complex-valued subanalytic function.

Remark 2.4
We will see in Section 7 that the elements of $\mathcal{C}^{\exp}_{\text{naive}}([0, +\infty))$ have certain *convergent* asymptotic expansions at $+\infty$. This implies that there are no Schwartz functions in $\mathcal{C}^{\exp}_{\text{naive}}([0, +\infty))$. In particular, the function $f(x) = \mathrm{e}^{-x}$ is not in $\mathcal{C}^{\exp}_{\text{naive}}([0, +\infty))$, while it can be easily shown that $f \in \mathcal{E}([0, +\infty))$. Now consider the function $\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,\mathrm{d}t$, which is clearly in $\mathcal{E}([0, +\infty))$. However, $\mathrm{Si}(x)$ is easily seen to have a *divergent* asymptotic expansion at $+\infty$; therefore, Si cannot be in $\mathcal{C}^{\exp}_{\text{naive}}([0, +\infty))$. (The details of the proof of this remark will be carried out in Section 7.)

This example suggests that, to construct $\mathcal{E}$, we cannot avoid including functions computable from single-variable integrals. Our main claim is that we will only need to consider single-variable integrals of the following special form.

Definition 2.5
For each $\ell \in \mathbb{N}$, subanalytic set $X \subseteq \mathbb{R}^m$, and $h \in \mathcal{S}(X \times \mathbb{R})$ such that $\forall x \in X$, $t \mapsto h(x,t) \in L^1(\mathbb{R})$, define $\gamma_{h,\ell} : X \to \mathbb{C}$ by

$$\gamma_{h,\ell}(x) = \int_{\mathbb{R}} h(x,t)\big(\log|t|\big)^{\ell}\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t.$$

This definition makes sense, because for each $x \in X$, requiring that $t \mapsto h(x,t)$ is in $L^1(\mathbb{R})$ is equivalent to requiring that $t \mapsto h(x,t)(\log|t|)^{\ell}$ is in $L^1(\mathbb{R})$. This is easily justified using elementary calculus and expanding $t \mapsto h(x,t)$ as in Remark 3.1.

Remark 2.6
If $g \in \mathcal{S}(X)$, then sometimes it will be convenient to see g as a function of type $\gamma_{h,\ell}$. To see this, take $\ell = 0$ and $h(x,t) = \frac{1}{2}g(x)\chi(t)$, where $\chi(t)$ is the characteristic function of the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. In particular, the constant function 1 can be viewed as a function of type $\gamma_{h,\ell}$ (and for the rest of this article we will implicitly assume so).

Note that, for $x \in X$, $\gamma_{h,\ell}(x) = \int_{\mathbb{R}} \widetilde{h}(x,t)\,\mathrm{d}t$, where $\widetilde{h}(x,t) = h(x,t)(\log|t|)^{\ell}\mathrm{e}^{\mathrm{i}t}$ if $t \neq 0$ and $\widetilde{h}(x,0) = 0$. Since $\widetilde{h} \in \mathcal{C}^{\exp}_{\text{naive}}(X \times \mathbb{R})$ and $\mathcal{C}^{\exp}_{\text{naive}}(X \times \mathbb{R}) \subseteq \mathcal{E}(X \times \mathbb{R})$, we must have $\gamma_{h,\ell} \in \mathcal{E}(X)$. This leads us to the following definition.

Definition 2.7
For each subanalytic set $X \subseteq \mathbb{R}^m$, define $\mathcal{C}^{\exp}(X)$ to be the $\mathcal{C}^{\exp}_{\text{naive}}(X)$-module of functions on X generated by

$$\left\{\gamma_{h,\ell} : \ell \in \mathbb{N} \text{ and } h \in \mathcal{S}(X \times \mathbb{R}) \text{ with } t \mapsto h(x,t) \text{ in } L^1(\mathbb{R})\right\}.$$

We write

$$\mathcal{C}^{\exp} := \left(\mathcal{C}^{\exp}(X)\right)_{X \text{ is subanalytic}}.$$

Thus, $f \in \mathcal{C}^{\exp}(X)$ if and only if f can be written as a finite sum

$$f(x) = \sum_{j=1}^{J} f_j(x)\gamma_{h_j,\ell_j}(x), \quad \text{with } f_j \in \mathcal{C}^{\exp}_{\text{naive}}(X), h_j \in \mathcal{S}(X \times \mathbb{R}), \text{ and } \ell_j \in \mathbb{N},$$

where $\forall x \in X, t \mapsto h_j(x,t) \in L^1(\mathbb{R})$ for each j.

Remark 2.8
Note that $\mathcal{C}^{\exp}$ is stable under composition with subanalytic functions in the following sense: if $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are subanalytic sets, $G : Y \to X$ is a subanalytic map, and $f \in \mathcal{C}^{\exp}(X)$, then $f \circ G \in \mathcal{C}^{\exp}(Y)$.

For each subanalytic set $X \subseteq \mathbb{R}^m$, it is clear that $\mathcal{C}^{\exp}(X) \subseteq \mathcal{E}(X)$. Hence, our next task is to study the parametric integrals of functions $f \in \mathcal{C}^{\exp}(X \times \mathbb{R}^n)$.

Notation 2.9
Write $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_n)$ for the standard coordinates on $\mathbb{R}^{m+n}$. Define $\Pi_m : \mathbb{R}^{m+n} \to \mathbb{R}^m$ by $\Pi_m(x, y) = x$. For each set $D \subseteq \mathbb{R}^{m+n}$, define the fiber of D over x by

$$D_x = \left\{y \in \mathbb{R}^n : (x, y) \in D\right\}.$$

Definition 2.10
For any Lebesgue measurable function $f : D \to \mathbb{C}$ with $D \subseteq \mathbb{R}^{m+n}$ and $\Pi_m(D) = X$, define the *locus of integrability of f over X* by

$$\text{Int}(f, X) := \left\{x \in X : f(x, \cdot) \in L^1(D_x)\right\}.$$

Remark 2.11
Let $f \in \mathcal{C}^{\exp}(X \times \mathbb{R}^n)$, and suppose that $f(x, \cdot) \in L^1(\mathbb{R}^n)$ for all $x \in X$. To compute $F(x) = \int_{y \in \mathbb{R}^n} f(x, y)\, \mathrm{d}y$, one typically works by induction on n, using Fubini's theorem to express it as an iterated integral

$$\int_{\mathbb{R}^{n-1}} \Big(\int_{\mathbb{R}} f(x, y_1, \ldots, y_{n-1}, y_n) \, \mathrm{d}y_n \Big) \, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-1}.$$

But then one is confronted with the fact that

$$(x, y_1, \ldots, y_{n-1}) \longmapsto \int_{\mathbb{R}} f(x, y_1, \ldots, y_{n-1}, y_n) \, \mathrm{d}y_n \tag{8}$$

might not be defined on all of $X \times \mathbb{R}^{n-1}$; all we know is that (8) is defined for all $x \in X$ and almost all $(y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$. So in order to have a stable framework that considers (8) to be a "parameterized integral" as well, it is useful to consider the more general situation from the start where one drops the assumption that $f(x, y)$ is integrable in y for all $x \in X$, but one then additionally studies the locus of integrability of f over X (see Theorem 2.20).

We are now ready to state the main result of this article.

THEOREM 2.12 (Stability under integration)
Let $f \in \mathcal{C}^{\exp}(X \times \mathbb{R}^n)$ for some subanalytic set $X \subseteq \mathbb{R}^m$ and $n \in \mathbb{N}$. Then there exists $F \in \mathcal{C}^{\exp}(X)$ such that

$$F(x) = \int_{\mathbb{R}^n} f(x, y) \, \mathrm{d}y \quad \textit{for all } x \in \mathrm{Int}(f, X).$$

It is clear from the definition that the module $\mathcal{C}^{\exp}(X)$ is closed under addition, but it is not so apparent from the definition alone whether $\mathcal{C}^{\exp}(X)$ is closed under multiplication. That $\mathcal{C}^{\exp}(X)$ is a ring is in fact a consequence of our main result.

COROLLARY 2.13
For each subanalytic set X, $\mathcal{C}^{\exp}(X)$ is a ring.

Proof
For any $n \in \mathbb{N}$ and functions $\gamma_{h_1,\ell_1}, \ldots, \gamma_{h_n,\ell_n}$, writing $y = (y_1, \ldots, y_n)$ we have

$$\begin{aligned} \prod_{j=1}^{n} \gamma_{h_j,\ell_j}(x) &= \prod_{j=1}^{n} \Big(\int_{\mathbb{R}} h(x, y_j) \big(\log |y_j|\big)^{\ell_j} \mathrm{e}^{\mathrm{i} y_j} \, \mathrm{d}y_j \Big) \\ &= \int_{\mathbb{R}^n} \Big(\prod_{j=1}^{n} h(x, y_j) \big(\log |y_j|\big)^{\ell_j} \mathrm{e}^{\mathrm{i} y_j} \Big) \, \mathrm{d}y, \end{aligned}$$

which is in $\mathcal{C}^{\exp}(X)$ by Theorem 2.12. It follows that $\mathcal{C}^{\exp}(X)$ is closed under multiplication. □

Remarks 2.14

(1) Theorem 2.12 and Corollary 2.13 imply that $\mathcal{C}^{\exp}$ is indeed the smallest collection of $\mathbb{C}$-algebras that contains $\mathcal{S} \cup \{e^{if} : f \in \mathcal{S}\}$ and that is stable under parametric integration. Hence, $\mathcal{C}^{\exp} = \mathcal{E}$.

(2) Note that $\mathcal{C}^{\exp}$ is closed under complex conjugation; hence, the real and imaginary parts of functions in $\mathcal{C}^{\exp}$ are also in $\mathcal{C}^{\exp}$. Moreover, $\mathcal{C}^{\exp}$ is closed under taking Fourier transforms (over $\mathbb{R}^m$ and over $\mathbb{R}$ with parameters, as in (5)).

(3) The ring $\mathcal{C}^{\exp}(X)$ can also be described as the smallest $\mathbb{C}$-algebra $\mathcal{A}(X)$ containing $\mathcal{C}(X)$ that is also stable by composition with subanalytic functions (the operation defined in Remark 2.8, where we take $n = m$) and by taking parametric Fourier transforms (the operation defined in (5)). To see this, note that Remark 2.8 and item (2) immediately above together imply that $\mathcal{A}(X) \subseteq \mathcal{C}^{\exp}(X)$. To prove the other inclusion, note first that a function of type $\gamma_{h,\ell}$ (as in Definition 2.5) is a parametric Fourier transform of a function in $\mathcal{C}(X)$. To see this, note that the parametric Fourier transform of the function $t \mapsto h(x,t)(\log t)^\ell$ is the function

$$F(x,y) = \int_{\mathbb{R}} h(x,t)(\log t)^\ell e^{-2\pi i t y}\, dt$$

and we have $\gamma_{h,\ell}(x) = F(x, -\frac{1}{2\pi})$, where evaluating F at the points $(x, -\frac{1}{2\pi})$ is allowed, thanks to the stability by composition with subanalytic functions. Moreover, the function $(x_1, \ldots, x_m) \mapsto e^{ix_1}$ belongs to $\mathcal{A}(X)$, since the functions $\frac{\sin x_1}{x_1}$ and $\frac{\cos x_1}{x_1}$ are Fourier transforms of the characteristic function of a suitable interval (see, e.g., [13]). Finally, by the stability under composition with subanalytic functions, if $\varphi \in \mathcal{S}(X)$, then $e^{i\varphi(x)} \in \mathcal{A}(X)$.

We now illustrate the main steps of the proof of Theorem 2.12.

Definition 2.15

Consider a subanalytic set X. Call $f : X \to \mathbb{C}$ a *generator* for $\mathcal{C}^{\exp}(X)$ if f is of the form

$$f(x) = g(x) e^{i\phi(x)} \gamma(x), \tag{9}$$

where $g \in \mathcal{C}(X)$, $\phi \in \mathcal{S}(X)$, and $\gamma = \gamma_{h,\ell}$ for some $\ell \in \mathbb{N}$ and $h \in \mathcal{S}(X \times \mathbb{R})$ with $t \mapsto h(x,t)$ in $L^1(\mathbb{R})$. When $\gamma = 1$, we will also call f a *generator* for $\mathcal{C}^{\exp}_{\text{naive}}(X)$. Note that a function is in $\mathcal{C}^{\exp}(X)$ if and only if the function can be expressed as a finite sum of generators for $\mathcal{C}^{\exp}(X)$, and likewise for $\mathcal{C}^{\exp}_{\text{naive}}(X)$.

Remark 2.16

The function f given in (9) is determined by the data (g, ϕ, h, ℓ). However, the choice of underlying data is not uniquely determined by the function f itself (see Remark 2.6). In what follows, we will always assume that, when we have a generator f, a choice of underlying data has been specified.

The purpose of the next two definitions is to identify a particular type of generator for $\mathcal{C}^{\exp}(X \times \mathbb{R}^n)$ which is integrable everywhere and whose integral can be computed using the Fubini–Tonelli theorem.

Definition 2.17
To the function (9) we associate the function $f^{\mathrm{abs}} \colon X \to [0, +\infty)$ defined by

$$f^{\mathrm{abs}}(x) := \left|g(x)\right| \gamma^{\mathrm{abs}}(x),$$

where $\gamma^{\mathrm{abs}} \colon X \to [0, +\infty)$ is defined by

$$\gamma^{\mathrm{abs}}(x) = \int_{\mathbb{R}} \left|h(x,t)\big(\log|t|\big)^{\ell}\right| \mathrm{d}t.$$

For f as in (9), note that for any $x \in X$ we have $|\gamma(x)| \le \gamma^{\mathrm{abs}}(x)$, so $|f(x)| \le f^{\mathrm{abs}}(x)$, and these inequalities can be strict. Observe that, for any given generator f for $\mathcal{C}^{\exp}(X)$, f^{abs} is uniquely determined by the underlying data used to define f as in (9), not by the function f itself.

Definition 2.18
We say that a generator f for $\mathcal{C}^{\exp}(X \times \mathbb{R}^n)$ is *superintegrable* over X if $f^{\mathrm{abs}}(x, \cdot) \in L^1(\mathbb{R}^n)$ for all $x \in X$.

In Section 4 we will prove the following result.

PROPOSITION 2.19 (Integration of superintegrable generators)
Let f be a generator for $\mathcal{C}^{\exp}(X \times \mathbb{R}^n)$ that is superintegrable over X, and define $F \colon X \to \mathbb{C}$ by

$$F(x) = \int_{\mathbb{R}^n} f(x,y)\,\mathrm{d}y.$$

Then $F \in \mathcal{C}^{\exp}(X)$.

The key step to the proof of Theorem 2.12 is given by the following interpolation result, which holds whenever we integrate with respect to a single variable $y \in \mathbb{R}$.

This result also gives a structure theorem for the locus of integrability of functions in $\mathcal{C}^{\exp}(X \times \mathbb{R})$.

THEOREM 2.20 (Interpolation and locus)
Let $f \in \mathcal{C}^{\exp}(X \times \mathbb{R})$ for some subanalytic set $X \subseteq \mathbb{R}^m$. Then there exists $g \in \mathcal{C}^{\exp}(X \times \mathbb{R})$ such that

$$\mathrm{Int}(g, X) = X$$

and

$$f(x, y) = g(x, y) \quad \textit{for all } x \in \mathrm{Int}(f, X) \textit{ and all } y \in \mathbb{R}.$$

Moreover, g can be written as a finite sum of generators for $\mathcal{C}^{\exp}(X \times \mathbb{R})$ that are superintegrable over X. Finally, there exists $h \in \mathcal{C}^{\exp}(X)$ such that

$$\mathrm{Int}(f, X) = \big\{x \in X : h(x) = 0\big\}.$$

Once we have established Theorem 2.20, the proof of Theorem 2.12 follows easily. The case $n = 1$ is implied by Theorem 2.20 and Proposition 2.19. For $n > 1$ we will use Fubini's theorem and induction on the number of variables with respect to which we integrate, as explained below.

Notation 2.21
Write $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ for coordinates on $\mathbb{R}^{m+n}$. For each $k \in \{1, \ldots, n\}$ and $\square \in \{<, >, \leq, \geq\}$, write $y_{\square k}$ for $(y_j)_{j \square k}$. For example, $y_{<k} = (y_1, \ldots, y_{k-1})$ and $y_{\leq k} = (y_1, \ldots, y_k)$, and also $\Pi_k(y) = y_{\leq k}$ and $\Pi_{m+k}(x, y) = (x, y_{\leq k})$.

Proof of Theorem 2.12
If $n = 1$, then by Theorem 2.20 there exists $g \in \mathcal{C}^{\exp}(X \times \mathbb{R})$ such that

$$f(x, y) = g(x, y) \quad \text{for all } x \in \mathrm{Int}(f, X) \text{ and all } y \in \mathbb{R}.$$

Moreover, g is a finite sum of superintegrable generators. The sum of their integrals belongs to $\mathcal{C}^{\exp}(X)$, thanks to Proposition 2.19, and gives us the required F.

Let $n > 1$. By Fubini's theorem, for all $x \in \mathrm{Int}(f, X)$, the function $g_x : y_{<n} \mapsto \int_{\mathbb{R}} f(x, y)\,\mathrm{d}y_n$ is defined for all $y_{<n}$ belonging to some set $E_x \subseteq \mathbb{R}^{n-1}$ such that the set $\mathbb{R}^{n-1} \setminus E_x$ has measure zero. Moreover, g_x is integrable with respect to $y_{<n}$, and $\int_{\mathbb{R}^n} f(x, y)\,\mathrm{d}y = \int_{E_x} g_x(y_{<n})\,\mathrm{d}y_{<n}$. If we apply the case $n = 1$ just proved to the function f seen as an element of $\mathcal{C}^{\exp}(\widetilde{X} \times \mathbb{R})$, where $\widetilde{X} = X \times \mathbb{R}^{n-1}$, then we obtain the existence of $F_1 \in \mathcal{C}^{\exp}(\widetilde{X})$ such that, $\forall (x, y_{<n}) \in \mathrm{Int}(f, \widetilde{X})$, $F_1(x, y_{<n}) =$

$\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y_n$. So, in particular,

$$\int_{\mathbb{R}^{n-1}} F_1(x, y_{<n})\,\mathrm{d}y_{<n} = \int_{E_x} g_x(y_{<n})\,\mathrm{d}y_{<n}$$

for all $x \in \mathrm{Int}(f, X)$. By the inductive hypothesis applied to F_1, we obtain the existence of $F \in \mathcal{C}^{\exp}(X)$ such that, $\forall x \in \mathrm{Int}(F_1, X)$, $F(x) = \int_{\mathbb{R}^{n-1}} F_1(x, y_{<n})\,\mathrm{d}y_{<n}$. Note that this argument shows that $\mathrm{Int}(f, X) \subseteq \mathrm{Int}(F_1, X)$; hence, we are done. □

The structure of the article is the following. In Section 3, we establish some notation and we review a series of known results about subanalytic and constructible functions. Such results are mainly due to [8], [20], and [9]. In Section 4 we prove Proposition 2.19.

Section 5 is the core of the article. In this section we prove a preparation theorem for functions in $\mathcal{C}^{\exp}(X \times \mathbb{R})$, namely, Theorem 5.2. This states that for each f there is a partition of $X \times \mathbb{R}$ into finitely many subanalytic sets such that, on each of these sets, f can be written as a finite sum of generators, each of which is either superintegrable or "naive in the last variable" (see Definition 5.1). As a consequence of the proof of this theorem we obtain that the functions in $\mathcal{C}^{\exp}$ are piecewise analytic (see Remark 5.8).

In Section 6 we complete the proof of Theorem 2.20. In order to do this, we apply Theorem 5.2. Subsequently, we show that any nonzero linear combination of nonintegrable generators for $\mathcal{C}^{\exp}_{\mathrm{naive}}(\mathbb{R})$ such that the arguments of the exponentials are distinct polynomials cannot be integrable (see Proposition 6.5(3)). The proof of this latter result uses the theory of continuously uniformly distributed maps and is postponed to Section 6.

Finally, in Section 7 we deduce a series of consequences of our main results: we prove an asymptotic result for elements of $\mathcal{C}^{\exp}_{\mathrm{naive}}(\mathbb{R})$, we give two examples of functions that are in $\mathcal{C}^{\exp}(\mathbb{R})$ but not in $\mathcal{C}^{\exp}_{\mathrm{naive}}(\mathbb{R})$, and we prove that $\mathcal{C}^{\exp}$ is stable under taking pointwise limits and also has an analogue for parametric families of the completeness theorem for L^p-spaces. Moreover, we prove that the extension of the Fourier transform to $L^2(\mathbb{R}^n)$ sends $\mathcal{C}^{\exp}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ onto $\mathcal{C}^{\exp}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

For the reader's convenience, we describe the dependence relations between the results in the two following diagrams. The first diagram concerns the stability of $\mathcal{C}^{\exp}$ under integration (see Theorem 2.12).

$$\begin{tikzcd}
\boxed{\text{Proposition 6.5}} \arrow[d] & \boxed{\text{Theorem 5.2}} \arrow[dl] \\
\boxed{\text{Theorem 2.20}} \arrow[d] & \boxed{\text{Proposition 2.19}} \arrow[dl] \\
\boxed{\text{Theorem 2.12}} &
\end{tikzcd}$$

The second diagram concerns the L^p-completeness and the stability of $\mathcal{C}^{\exp}$ under the Fourier–Plancherel transform (see Proposition 8.2 and Theorem 8.3).

$$\begin{tikzcd}
\boxed{\text{Remark 8.5}} \arrow[dr] & \boxed{\text{Lemma 8.9}} \arrow[d] & \boxed{\text{Proposition 8.7}} \arrow[l] & \boxed{\text{Lemma 8.6}} \arrow[l] \\
\boxed{\text{Theorem 5.2}} \arrow[dr] & \boxed{\text{Lemma 8.10}} \arrow[d] & & \\
 & \boxed{\text{Proposition 8.2}} \arrow[d] & & \\
 & \boxed{\text{Theorem 8.3}} & &
\end{tikzcd}$$

3. Preparation of subanalytic and constructible functions

This section gives the version of the preparation theorem for subanalytic and constructible functions that we will use throughout the article. It is mostly a review of ideas from [20] and [9] but formulated in a way that is convenient for our current purposes.

Remark 3.1

It is well known that every subanalytic function of one variable admits a convergent

Puiseux expansion at $+\infty$ (see, e.g., [22], [28]). More precisely, if $g \in \mathcal{S}(\mathbb{R})$, then there are $c \in \mathbb{R}$, $d \in \mathbb{N}$, $r \in \mathbb{Q}$ (which can be chosen as an integer multiple of $\frac{1}{d}$), and an absolutely convergent power series $H \in \mathbb{R}\{y\}$, with $H(0) = 0$, such that for x sufficiently large

$$g(x) = cx^r\big(1 + H(x^{-\frac{1}{d}})\big). \tag{10}$$

In particular, for x large, g can be written as

$$g(x) = p(x^{\frac{1}{d}}) + g_0(x), \tag{11}$$

where $p \in \mathbb{R}[y]$, with $p(0) = 0$, and g_0 is a *bounded* subanalytic function.

The subanalytic preparation theorem given in [20, Théorème 1] can be viewed as a parametric version (in several variables) of the preceding remark, and the constructible preparation theorem given in [9, Corollary 3.5] is the natural extension of this latter result to the context of constructible functions.

We fix some notation.

Definition 3.2
A set $A \subseteq \mathbb{R}^{m+n}$ is *open over* $\mathbb{R}^m$ if the fiber A_x is open in $\mathbb{R}^n$ for all $x \in \Pi_m(A)$. For any set $X \subseteq \mathbb{R}^m$, call a map $f : X \to \mathbb{R}^n$ *analytic* if f extends to an analytic map on a neighborhood of X in $\mathbb{R}^m$.

Recall Notation 2.21.

Definition 3.3
A set $A \subseteq \mathbb{R}^{m+n}$ is a *cell over* $\mathbb{R}^m$ if A is subanalytic and, for each $j \in \{1, \ldots, n\}$, $\Pi_{m+j}(A)$ is either the graph of an analytic function in $\mathcal{S}(\Pi_{m+j-1}(A))$ or else

$$\Pi_{m+j}(A) = \big\{(x, y_{\le j}) : (x, y_{<j}) \in \Pi_{m+j-1}(A), a_j(x, y_{<j}) < y_j < b_j(x, y_{<j})\big\}$$

for some analytic, subanalytic functions $a_j(x, y_{<j}) < b_j(x, y_{<j})$, where we also allow the possibility that $a_j \equiv -\infty$ and the possibility that $b_j \equiv +\infty$. If $m = 0$, we will just say that A is a *subanalytic cell*.

Definition 3.4
Let $A \subseteq \mathbb{R}^{m+1}$ be a cell over $\mathbb{R}^m$ that is open over $\mathbb{R}^m$, and write $(x, y) = (x_1, \ldots, x_m, y)$ for coordinates on $\mathbb{R}^{m+1}$. Call $\theta : \Pi_m(A) \to \mathbb{R}$ a *center* for A if the following hold.

(1) θ is an analytic subanalytic function.

(2) The graph of θ is disjoint from A and is either contained in or is disjoint from the closure of A in $\Pi_m(A) \times \mathbb{R}$.

(3) The image of θ is contained in one of the sets $(-\infty, 0)$, $\{0\}$, or $(0, +\infty)$. Moreover, when $\theta \neq 0$, the closure of $\{|y/\theta(x)| : (x, y) \in A\}$ in $\mathbb{R}$ is a compact subset of $(0, +\infty)$.

(4) The set $\{y - \theta(x) : (x, y) \in A\}$ is contained in one of the sets $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, or $(1, +\infty)$.

Note that, when θ is a center for A, there exist unique $\sigma, \tau \in \{-1, 1\}$ such that

$$\left\{\sigma\left(y - \theta(x)\right)^{\tau} : (x, y) \in A\right\} \subseteq (1, +\infty)$$

and A is of the form

$$A = \left\{(x, y) : x \in \Pi_m(A), a(x) < \sigma\left(y - \theta(x)\right)^{\tau} < b(x)\right\} \tag{12}$$

for some analytic, subanalytic functions $1 \leq a(x) < b(x)$, where either $b < +\infty$ on $\Pi_m(A)$ or $b \equiv +\infty$ on $\Pi_m(A)$.

Define $P_\theta = (P_{\theta,1}, \ldots, P_{\theta,m+1})\colon \Pi_m(A) \times (1, +\infty) \to \Pi_m(A) \times \mathbb{R}$ by

$$P_\theta(x, y) = \left(x, \sigma y^{\tau} + \theta(x)\right). \tag{13}$$

Define $A_\theta = P_\theta^{-1}(A)$, and note that

$$A_\theta = \left\{(x, y) : x \in \Pi_m(A), a(x) < y < b(x)\right\} \tag{14}$$

and that P_θ restricts to a bijection $P_\theta\colon A_\theta \to A$ whose inverse is given by

$$P_\theta^{-1}(x, y) = \left(x, \sigma\left(y - \theta(x)\right)^{\tau}\right).$$

We will henceforth restrict P_θ to A_θ, considering it to be a bijection from A_θ to A.

Remark 3.5

When $b \equiv +\infty$, necessarily $\tau = 1$ and the second sentence of Definition 3.4(3) implies that $\theta = 0$.

For any polyradius $r = (r_1, \ldots, r_N) \in (0, +\infty)^N$, define

$$B_r(\mathbb{C}) = \left\{z \in \mathbb{C}^N : |z_1| \leq r_1, \ldots, |z_N| \leq r_N\right\} \qquad \text{and}$$
$$B_r(\mathbb{R}) = B_r(\mathbb{C}) \cap \mathbb{R}^N,$$

where $z = (z_1, \ldots, z_N)$.

Definition 3.6
Let $\psi: X \to \mathbb{R}^N$ be a subanalytic map such that $\psi(X) \subseteq B_r(\mathbb{R})$ for some $r \in (0,+\infty)^N$. We call $f: X \to \mathbb{R}$ a *ψ-function* if there exists a real analytic function F such that $f = F \circ \psi$ and F is given by a single convergent power series in N variables, centered at 0 and converging in some open neighborhood of $B_r(\mathbb{R})$ in $\mathbb{R}^N$.

Observe that F extends uniquely to a complex analytic function on a neighborhood of $B_r(\mathbb{C})$ in $\mathbb{C}^N$. If we additionally have that

$$\left|F(z) - 1\right| < 1 \quad \text{for all } z \in B_r(\mathbb{C}),$$

then we call f a *ψ-unit*.

Remarks 3.7
Let $f = F \circ \psi$ be a ψ-unit, with r as above. Then we have the following.

(1) There exist strictly positive constants $k < K$ such that $k < |F(x)| < K$ for every $x \in B_r(\mathbb{R})$.

(2) The set $F(B_r(\mathbb{C}))$ is compact. Therefore, there exists $\varepsilon \in (0,1)$ such that $|F(z) - 1| < 1 - \varepsilon$ for all $z \in B_r(\mathbb{C})$.

(3) Remark 3.5 shows that the natural logarithm extends to a holomorphic function on a neighborhood of $F(B_r(\mathbb{C}))$ in $\mathbb{C}^N$, so $\log F$ is given by a single convergent power series on $B_r(\mathbb{C})$ centered at 0. Therefore, $\log f : X \to \mathbb{R}$ is a ψ-function.

Definition 3.8
Consider the cell A in (12) and a bounded, analytic, subanalytic map ψ, defined on A, of the form

$$\psi(x,y) = \Big(c_1(x), \ldots, c_N(x), \Big(\frac{a(x)}{\sigma(y-\theta(x))^\tau}\Big)^{1/d}, \Big(\frac{\sigma(y-\theta(x))^\tau}{b(x)}\Big)^{1/d}\Big)$$

$$\text{if } b < +\infty,$$

$$\psi(x,y) = \Big(c_1(x), \ldots, c_N(x), \Big(\frac{a(x)}{\sigma(y-\theta(x))^\tau}\Big)^{1/d}\Big) \quad \text{if } b \equiv +\infty,$$

for some positive integer d and some analytic functions $c_1, \ldots, c_N$.

We say that a subanalytic function $f: A \to \mathbb{R}$ is *ψ-prepared* if

$$f(x,y) = f_0(x)\left|y - \theta(x)\right|^\nu u(x,y)$$

on A for some analytic $f_0 \in \mathcal{S}(\Pi_m(A))$, $\nu \in \mathbb{Q}$, and u a ψ-unit.

Remark 3.9
We will frequently apply this concept to the situation when $A = A_\theta$ (namely, $\theta = 0$ and $\sigma = \tau = 1$), in which case

$$\begin{aligned} \psi(x,y) &= \Big(c_1(x),\ldots,c_N(x),\Big(\frac{a(x)}{y}\Big)^{1/d},\Big(\frac{y}{b(x)}\Big)^{1/d}\Big) \quad \text{if } b < +\infty, \\ \psi(x,y) &= \Big(c_1(x),\ldots,c_N(x),\Big(\frac{a(x)}{y}\Big)^{1/d}\Big) \quad \text{if } b \equiv +\infty, \end{aligned} \tag{15}$$

and

$$f(x,y) = f_0(x)y^\nu u(x,y), \tag{16}$$

on A_θ for some analytic $f_0 \in \mathcal{S}(\Pi_m(A))$, $\nu \in \mathbb{Q}$, and u a ψ-unit.

PROPOSITION 3.10 (Preparation of constructible functions)
Let $D \subseteq \mathbb{R}^{m+1}$ be subanalytic, and let $\mathcal{F} \subseteq \mathcal{C}(D)$ be a finite set of constructible functions. Then there exists a finite partition $\mathcal{A}$ of D into cells over $\mathbb{R}^m$ such that for each $A \in \mathcal{A}$ that is open over $\mathbb{R}^m$ there exists a center θ for A such that, for each $f \in \mathcal{F}$, we can write $f \circ P_\theta$ as a finite sum

$$f \circ P_\theta(x,y) = \sum_{j\in J} g_j(x)y^{r_j}(\log y)^{s_j}h_j(x,y) \tag{17}$$

on A_θ, where
(1) *A_θ is as in (14);*
(2) *P_θ is as in (13);*
(3) *the functions h_j are ψ-functions (see Definition 3.6), where ψ is as in (15) for some analytic functions $c_1,\ldots,c_N$ and some integer $d > 0$;*
(4) *$s_j \in \mathbb{N}$ and the r_j's are integer multiples of $1/d$;*
(5) *the functions g_j are analytic and in $\mathcal{C}(\Pi_m(A))$.*

Proof
We apply [9, Corollary 3.5], and we obtain a cell decomposition $\mathcal{A}$ such that (17) holds, with conditions (1) and (2) satisfied. Up to refining $\mathcal{A}$, we may assume that (5) also holds. We must now show that, up to some refinement of $\mathcal{A}$, we may assume that conditions (3) and (4) hold as well. By [9, Corollary 3.5], we know that a weaker version of condition (3) holds, namely, the h_j's are of the form $\widetilde{F_j} \circ \widetilde{\psi}$, where $\widetilde{F_j}$ is a power series converging on some open set O_j containing the closure of the image of $\widetilde{\psi}$ and $\widetilde{\psi}$ is a bounded map whose components are

$$c_1(x),\ldots,c_M(x),\big(e_1(x)/y\big)^{1/d},\big(e_2(x)y\big)^{1/d}$$

for some $M \geq 0$, some $d > 0$, and some analytic, subanalytic functions $c_1, \ldots, c_M$, e_1, e_2. We now explain, in the case in which $b(x) < +\infty$, how we can obtain the quotients $a(x)/y$ and $y/b(x)$ as arguments instead of $e_1(x)/y$ and $e_2(x)y$. (The case $b(x) = +\infty$ is similar and even easier.) Since $e_1(x)/y$ and $e_2(x)y$ are bounded and since y runs from $a(x)$ to $b(x)$, one has that $e_1(x)/a(x)$ and $e_2(x)b(x)$ are also bounded. Let $\psi(x, y)$ be

$$\Big(c_1(x), \ldots, c_M(x), \big(e_1(x)/a(x)\big)^{1/d}, \big(e_2(x)b(x)\big)^{1/d}, \Big(\frac{a(x)}{y}\Big)^{1/d}, \Big(\frac{y}{b(x)}\Big)^{1/d}\Big)$$

and

$$\pi : \mathbb{R}^{M+4} \ni (z_1, \ldots, z_M, Z_1, Z_2, Z_3, Z_4) \mapsto (z_1, \ldots, z_M, Z_1 Z_3, Z_2 Z_4) \in \mathbb{R}^{M+2}.$$

Then ψ is bounded, $\widetilde{\psi} = \pi \circ \psi$, and the closure of the image of ψ is contained in the open sets $\pi^{-1}(O_j)$. We rename $c_{M+1} = (e_1(x)/a(x))^{1/d}$ and $c_{M+2} = (e_2(x)b(x))^{1/d}$, and we set $N = M + 2$. It is clear that the power series $F_j = \widetilde{F_j} \circ \pi$ converge on $\pi^{-1}(O_j)$ and that $F_j(\psi) = \widetilde{F_j}(\widetilde{\psi})$ on A_θ. Hence, condition (3) holds. Finally, by replacing d by an integer multiple if necessary, we can assume that condition (4) also holds. $\square$

Remark 3.11
We have stated Proposition 3.10 in the transformed coordinates (via P_θ) out of convenience. In the original coordinates, (17) becomes

$$f(x, y) = \sum_{j \in J} \widetilde{g}_j(x) \big|y - \theta(x)\big|^{\widetilde{r}_j} \big(\log\big|y - \theta(x)\big|\big)^{s_j} \widetilde{h}_j(x, y)$$

on A, where $\widetilde{g}_j(x) = \tau^{s_j} g_j(x)$, $\widetilde{r}_j = \tau r_j$, and $\widetilde{h}(x, y) = h \circ P_\theta^{-1}(x, y)$.

Remark 3.12
If $\mathcal{F} \subseteq \mathcal{S}(D)$ is a finite collection of subanalytic functions, then the proof of Proposition 3.10 (where we replace the use of [9, Corollary 3.5] by the use of [9, Theorem 3.4]) shows that, for each $f \in \mathcal{F}$, on A_θ we can write $f \circ P_\theta$ in the ψ-prepared form in (16). In addition, it follows from the proof of the subanalytic preparation [19, Théorème 1] that if $\varepsilon \in (0, 1)$ is given beforehand, then the preparation can be constructed so that each ψ-unit u, as given in (16), is within ε of 1, by which we mean that $u = U \circ \psi$ for some $r \in (0, \infty)^M$ (where $M = N + 2$ when $b < +\infty$, and $M = N + 1$ when $b \equiv +\infty$) such that $\psi(A_\theta) \subseteq B_r(\mathbb{R})$ and some real analytic function U on $B_r(\mathbb{R})$ that extends to a complex analytic function on a neighborhood of $B_r(\mathbb{C})$ in $\mathbb{C}^M$ such that $|U(z) - 1| < \varepsilon$ for all $z \in B_r(\mathbb{C})$.

Remark 3.13
In the situation described in Proposition 3.10, we may also assume that the following two properties hold when $b \equiv +\infty$. Let $J_1 = \{j \in J : h_j = 1\}$. Then,

(1) for each $j \in J \setminus J_1$, $r_j < -1$;

(2) $((r_j, s_j))_{j \in J_1}$ is a family of distinct pairs in $\mathbb{Q} \times \mathbb{N}$.

To see this, note that because $b \equiv +\infty$, we may write h_j as a convergent power series

$$h_j(x, y) = \sum_{k=0}^{+\infty} h_{j,k}(x) \Big(\frac{a(x)}{y}\Big)^{k/d}$$

for $(c_1, \ldots, c_N)$-functions $h_{j,k}$. To obtain property (1), for each $j \in J$, fix $n_j \in \mathbb{N}$ such that

$$r_j - \frac{n_j}{d} < -1,$$

and write the jth term of (17) as

$$\sum_{k=0}^{n_j - 1} g_j(x) h_{j,k}(x) a(x)^{k/d} y^{r_j - k/d} (\log y)^{s_j} + R_j(x, y),$$

where

$$R_j(x, y) = g_j(x) a(x)^{n_j/d} y^{r_j - n_j/d} (\log y)^{s_j} \Big(\sum_{k=n_j}^{+\infty} h_{j,k}(x) \Big(\frac{a(x)}{y}\Big)^{(k-n_j)/d} \Big).$$

To obtain property (2), simply sum up terms in (17) for $j \in J_1$ with equal powers r_j and s_j.

We now study the integrability properties of the prepared form given in (17). The following remarks will be useful in Sections 4 and 6.

Remark 3.14
Consider the situation described in Proposition 3.10 for some $A \in \mathcal{A}$. In the notation of Proposition 3.10, for each $j \in J$, write

$$G_j(x, y) := g_j(x) y^{r_j} (\log y)^{s_j} h_j(x, y).$$

(1) Note that we have

$$\partial_y P_\theta(y) := \frac{\partial P_{\theta,m+1}}{\partial y}(x, y) = \sigma \tau y^{\tau - 1},$$

that $\tau - 1$ equals either 0 or -2, and that

$$\mathrm{Int}\big(f \restriction A, \Pi_m(A)\big) = \mathrm{Int}\big((f \circ P_\theta) \partial_y P_\theta, \Pi_m(A)\big).$$

(2) For each $j \in J$ and $x \in \Pi_m(A)$, $y \mapsto G_j(x, y)$ extends to a continuous (in fact, analytic) function on the closure in $\mathbb{R}$ of the fiber $(A_\theta)_x$, and likewise for $\partial_y P_\theta$.
In particular, when $b < +\infty$, $\mathrm{Int}(G_j \partial_y P_\theta, \Pi_m(A)) = \Pi_m(A)$ for each $j \in J$.

(3) Let $b \equiv +\infty$, and recall Remark 3.13(1). For each $j \in J \setminus J_1$ and $x \in \Pi_m(A)$, the function $y \mapsto G_j(x, y)\partial_y P_\theta(y)$ is $o(y^{\tau-2})$ as $y \to +\infty$ and is therefore integrable. For each $j \in J_1$,

$$G_j(x, y)\partial_y P_\theta(y) = \sigma\tau g_j(x) y^{r_j+\tau-1}(\log y)^{s_j},$$

which is integrable in y if and only if $g_j(x) = 0$ or $r_j + \tau < 0$.
Therefore, by defining

$$J^{\mathrm{Int}} = (J \setminus J_1) \cup \{j \in J_1 : r_j + \tau < 0\},$$

we see that, for each $j \in J$,

$$\mathrm{Int}\big(G_j \partial_y P_\theta, \Pi_m(A)\big) = \begin{cases} \Pi_m(A) & \text{if } j \in J^{\mathrm{Int}}, \\ \{x \in \Pi_m(A) : g_j(x) = 0\} & \text{if } j \in J \setminus J^{\mathrm{Int}}. \end{cases}$$

(4) In the situation of Remark 3.13, define the constructible functions

$$g(x, y) = \sum_{j \in J^{\mathrm{Int}}} G_j(x, y) \quad \text{for all } (x, y) \in A_\theta,$$
$$h(x) = \sum_{j \in J \setminus J^{\mathrm{Int}}} g_j^2(x) \quad \text{for all } x \in \Pi_m(A).$$

Then

$$\mathrm{Int}\big(f \restriction A, \Pi_m(A)\big) = \big\{x \in \Pi_m(A) : h(x) = 0\big\},$$
$$\mathrm{Int}\big(g\partial_y P_\theta, \Pi_m(A)\big) = \Pi_m(A),$$

and

$$f \circ P_\theta(x, y) = g(x, y) \quad \text{for all } (x, y) \in A_\theta \text{ with } x \in \mathrm{Int}\big(f \restriction A, \Pi_m(A)\big).$$

To see this, note that Remark 3.13 shows that $\mathrm{Int}(g\partial_y P_\theta, \Pi_m(A)) = \Pi_m(A)$, and clearly

$$f \circ P_\theta = g \quad \text{on the set } \big\{(x, y) \in A_\theta : h(x) = 0\big\},$$

so $\{x \in \Pi_m(A) : h(x) = 0\} \subseteq \mathrm{Int}(f \restriction A, \Pi_m(A))$. To show that $\mathrm{Int}(f \restriction A, \Pi_m(A)) \subseteq \{x \in \Pi_m(A) : h(x) = 0\}$, note that if $x \in \Pi_m(A)$ is such that

$h(x) \neq 0$, then by choosing j_0 in the set $\{j \in J \setminus J^{\mathrm{Int}} : g_j(x) \neq 0\}$ with (r_{j_0}, s_{j_0}) greatest with respect to the lexicographical order on $\mathbb{Q} \times \mathbb{R}$, it follows from Remark 3.13(2) that

$$\lim_{y \to +\infty} \frac{f(x,y)}{G_{j_0}(x,y)} = 1,$$

so $f(x,\cdot) \notin L^1(A_x)$ by the remark.

(5) In particular, if $\mathrm{Int}(f, X) = X$, then Remarks 3.14(2) and 3.14(4) above show that, for each $j \in J$, we have $\mathrm{Int}(G_j \partial_y P_\theta, \Pi_m(A)) = \Pi_m(A)$.

4. Integrating superintegrable generators

This section is dedicated to the proof of Proposition 2.19, of which we recall the statement.

PROPOSITION

Let f be a generator for $\mathcal{C}^{\exp}(X \times \mathbb{R}^n)$ that is superintegrable over X, and define $F : X \to \mathbb{C}$ by

$$F(x) = \int_{\mathbb{R}^n} f(x,y)\,\mathrm{d}y.$$

Then $F \in \mathcal{C}^{\exp}(X)$.

Proof

Assume that $X \subseteq \mathbb{R}^m$, and write

$$f(x,y) = g(x,y)\mathrm{e}^{\mathrm{i}\phi(x,y)}\gamma(x,y) \quad \text{for } (x,y) \in X \times \mathbb{R}^n,$$

where $g \in \mathcal{C}(X \times \mathbb{R}^n)$, $\phi \in \mathcal{S}(X \times \mathbb{R}^n)$, and $\gamma = \gamma_{h,\ell}$ for some $\ell \in \mathbb{N}$ and $h \in \mathcal{S}(X \times \mathbb{R}^n \times \mathbb{R})$ with $\mathrm{Int}(h, X \times \mathbb{R}^n) = X \times \mathbb{R}^n$.

Because $|f(x,y)| \leq f^{\mathrm{abs}}(x,y)$ for all $(x,y) \in X \times \mathbb{R}^n$ (see Definition 2.17), it follows that $f(x,\cdot) \in L^1(\mathbb{R}^n)$ for all $x \in X$. Moreover, the Fubini–Tonelli theorem shows that, for each $x \in X$,

$$(y,t) \mapsto g(x,y)h(x,y,t)\big(\log|t|\big)^{\ell}$$

is in $L^1(\mathbb{R}^n \times \mathbb{R})$, and the iterated integral

$$\int_{\mathbb{R}^n} f(x,y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}} g(x,y)\mathrm{e}^{\mathrm{i}\phi(x,y)} h(x,y,t)\big(\log|t|\big)^{\ell} \mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t \Big)\,\mathrm{d}y$$

can be computed as a product integral

$$\int_{\mathbb{R}^n \times \mathbb{R}} g(x,y)\mathrm{e}^{\mathrm{i}\phi(x,y)} h(x,y,t)\big(\log|t|\big)^{\ell} \mathrm{e}^{\mathrm{i}t}\, \mathrm{d}y \wedge \mathrm{d}t.$$

Therefore, up to replacing n by $n+1$, we may simply assume that $\gamma = 1$.

Now construct a finite partition $\mathcal{A}$ of $X \times \mathbb{R}^n$ into cells over $\mathbb{R}^m$ such that, for each $A \in \mathcal{A}$ that is open over $\mathbb{R}^m$, either $\phi(x,y) = \phi_0(x)$ on A for some $\phi_0 \in \mathcal{S}(\Pi_m(A))$ or else the function $y \mapsto \phi(x,y)$ is C^1 on A_x with $\sigma \frac{\partial \phi}{\partial y_j} > 0$ on A_x for some $\sigma \in \{-1,1\}$ and $j \in \{1,\ldots,n\}$.

When $\phi = \phi_0$,

$$\int_{A_x} f(x,y)\,\mathrm{d}y = \mathrm{e}^{\mathrm{i}\phi_0(x)} \int_{A_x} g(x,y)\,\mathrm{d}y. \tag{18}$$

The fact that $\mathcal{C}$ is stable under integration (see [8], [9]) shows that the integral of g with respect to y is in $\mathcal{C}(\Pi_m(A))$. Hence, (18) is in $\mathcal{C}^{\exp}_{\text{naive}}(\Pi_m(A))$.

In the other case, by pulling back by the inverse of the map $(x,y) \mapsto (x, \phi(x,y), y_{<j}, y_{>j})$ and multiplying by the Jacobian of this map, we may simply assume that $\phi(x,y) = y_1$. Write $\widetilde{A} = \Pi_{m+1}(A)$, and note that the function $\widetilde{g}\colon \widetilde{A} \to \mathbb{R}$ defined by

$$\widetilde{g}(x,y_1) = \int_{A_{(x,y_1)}} g(x,y_1,y_{>1})\,\mathrm{d}y_{>1} \quad \text{for each } (x,y_1) \in \widetilde{A},$$

is constructible and that

$$\int_{A_x} g(x,y)\mathrm{e}^{\mathrm{i}\phi(x,y)}\,\mathrm{d}y = \int_{\widetilde{A}_x} \widetilde{g}(x,y_1)\mathrm{e}^{\mathrm{i}y_1}\,\mathrm{d}y_1 \quad \text{for each } x \in \Pi_m(A).$$

We apply Proposition 3.10 to $\widetilde{g}(x,y_1)$ and then work piecewise, thereby focusing on one open cell $\widetilde{B} \subseteq \widetilde{A}$ given by the preparation, which is open over $\mathbb{R}^m$. By applying Remark 3.14(5), we may write

$$\int_{\widetilde{B}_x} \widetilde{g}(x,y_1)\mathrm{e}^{\mathrm{i}y_1}\,\mathrm{d}y_1$$

as a finite sum of terms of the form

$$g_0(x) \int_{\widetilde{B}_x} |y_1 - \theta(x)|^r \widetilde{u}(x,y_1)\big(\log|y_1 - \theta(x)|\big)^s \mathrm{e}^{\mathrm{i}y_1}\,\mathrm{d}y_1, \tag{19}$$

where $g_0 \in \mathcal{C}(\Pi_m(\widetilde{B}))$, $r \in \mathbb{Q}$, $s \in \mathbb{N}$, $\widetilde{u}$ is a ψ-function (for some ψ), and θ is the center given by the preparation on $\widetilde{B}$. Thus, for some $\sigma \in \{-1,1\}$, by applying the coordinate change $(x,y_1) \mapsto (x, \sigma y_1 + \theta(x))$ we may write (19) as

$$\sigma g_0(x)\mathrm{e}^{\mathrm{i}\theta(x)} \int_{B_x} y_1^r (\log y_1)^s u(x,y_1)\mathrm{e}^{\mathrm{i}\sigma y_1}\,\mathrm{d}y_1, \tag{20}$$

where $B_x \subseteq (0, +\infty)$ and u are the pullbacks of $\widetilde{B_x}$ and $\widetilde{u}$ by this coordinate change. Note that, up to performing the coordinate transformation $y_1 \mapsto \sigma y_1$, the one-variable integral in (20) is of the form $\gamma_{h,l}$ with $l = s$ and $h(x, y_1) = y_1^r u(x, y_1)\chi_{B_x}(y_1)$, where χ_{B_x} is the characteristic function of the subanalytic set B_x. This concludes the proof of Proposition 2.19. □

5. Preparation of functions in $\mathcal{C}^{\exp}$

Throughout this section, X denotes a subanalytic subset of $\mathbb{R}^m$, and we write (x, y) for coordinates on $\mathbb{R}^m \times \mathbb{R}$. This section states and proves our main preparation theorem for functions in $\mathcal{C}^{\exp}$. The purpose of the preparation theorem is to express a given $f \in \mathcal{C}^{\exp}(X \times \mathbb{R})$ as a finite sum of generators for $\mathcal{C}^{\exp}(X \times \mathbb{R})$ that are either superintegrable over X or are "naive in y" (in the sense that the γ-functions in these terms depend only on x and not on y; see Definition 5.1).

Definition 5.1

Let $A \subseteq \mathbb{R}^{m+1}$ be a subanalytic set, and let $T(x, y) \in \mathcal{C}^{\exp}(A)$ be a generator. We say that T is *naive* in y if T is of the form

$$T(x, y) = f(x) y^r (\log y)^s \mathrm{e}^{\mathrm{i}\phi(x,y)},$$

where $f \in \mathcal{C}^{\exp}(\Pi_m(A))$, $r \in \mathbb{Q}$, $s \in \mathbb{N}$, and $\phi \in \mathcal{S}(A)$. Note that, if T is naive in y, then the function γ appearing in (9) does not depend on y.

We use the notation from Definition 3.4 in the following theorem.

THEOREM 5.2

Let $X \subseteq \mathbb{R}^m$ be a subanalytic set, and let $f \in \mathcal{C}^{\exp}(X \times \mathbb{R})$. Then there exists a finite partition $\mathcal{A}$ of $X \times \mathbb{R}$ into cells over $\mathbb{R}^m$ such that, for each $A \in \mathcal{A}$ that is open over $\mathbb{R}^m$, there exists a center θ for A for which we may express $f \circ P_\theta$ as a finite sum

$$f \circ P_\theta(x, y) = \sum_{j \in J} T_j(x, y)$$

on $A_\theta = \{(x, y) : x \in \Pi_m(A), a(x) < y < b(x)\}$, where each T_j is a generator for $\mathcal{C}^{\exp}(A_\theta)$, such that

(1) *if $b < +\infty$, then for each j, T_j is superintegrable over $\Pi_m(A)$;*

(2) *if $b \equiv +\infty$, then there exists a positive integer d and a partition $J = J^{\mathrm{int}} \cup J^{\mathrm{naive}}$ such that*

  (a) *for each $j \in J^{\mathrm{int}}$, T_j is superintegrable over $\Pi_m(A)$;*

  (b) *for each $j \in J^{\mathrm{naive}}$, T_j is naive in y, is not superintegrable over $\Pi_m(A)$, and is of the form*

$$T_j(x,y) = f_j(x) y^{r_j} (\log y)^{s_j} \mathrm{e}^{\mathrm{i}\phi_j(x,y)}, \tag{21}$$

where $f_j \in \mathcal{C}^{\exp}(\Pi_m(A))$*,* $r_j \in \mathbb{Q} \cap [-1, +\infty)$*,* $s_j \in \mathbb{N}$*, and* ϕ_j *is a polynomial in* $y^{1/d}$ *(for some* $d \in \mathbb{N}$*) with coefficients in* $\mathcal{S}(\Pi_m(A))$ *such that* $\phi_j(x,0) = 0$ *for all* $x \in \Pi_m(A)$*; moreover,*

$$\big((r_j, s_j, \phi_j(x,y))\big)_{j \in J^{\text{naive}}}$$

is a family of distinct tuples in $\mathbb{Q} \times \mathbb{N} \times \mathbb{R}[y^{1/d}]$*.*

Remark 5.3
Let us restrict our attention to a cell of the form

$$A = \big\{(x,y) : x \in \Pi_m(A), y > a(x)\big\}. \tag{22}$$

(By Remark 3.5, we have $A = A_\theta$.) The proof of Theorem 5.2 will actually show that, for every $j \in J$, there are $r_j \in \mathbb{Q}$, $s_j \in \mathbb{N}$, and a function $g_j(x,y) \in \mathcal{C}^{\exp}(X \times \mathbb{R})$ which is bounded in y (more precisely, there is a subanalytic function $\eta : \Pi_m(A) \to [0, +\infty)$ such that, $\forall y > a(x)$, $|g_j(x,y)| < \eta(x)$) such that

$$T_j(x,y) = y^{r_j} (\log y)^{s_j} g_j(x,y). \tag{23}$$

Moreover, if $j \in J^{\text{Int}}$, then $r_j < -1$, and if $j \in J^{\text{naive}}$, then, in the notation of (21), we have $g_j(x,y) = f_j(x)\mathrm{e}^{\mathrm{i}\phi_j(x,y)}$.

The proof of Theorem 5.2 will be broken down into several propositions and lemmas.

Definition 5.4
Let $X \subseteq \mathbb{R}^m$ be a subanalytic set, and let $A \subseteq X \times \mathbb{R}$ be a cell over $\mathbb{R}^m$ which is open over $\mathbb{R}^m$. Let θ be a center for A, so that we can write

$$A_\theta = \big\{(x,y) : x \in \Pi_m(A), a(x) < y < b(x)\big\},$$

for some analytic, subanalytic functions $1 \leq a(x) < b(x)$, where we also allow the case when $b \equiv +\infty$ on $\Pi_m(A)$, as in Definition 3.4.

Fix $d \in \mathbb{N} \setminus \{0\}$ and a bounded, analytic, subanalytic map ψ on A_θ of the form

$$\psi(x,y) = \Big(c_1(x), \dots, c_N(x), \Big(\frac{a(x)}{y}\Big)^{1/d}, \Big(\frac{y}{b(x)}\Big)^{1/d}\Big) \quad \text{if } b < +\infty,$$

and $$\psi(x,y) = \Big(c_1(x), \dots, c_N(x), \Big(\frac{a(x)}{y}\Big)^{1/d}\Big) \quad \text{if } b \equiv +\infty. \tag{24}$$

Let J be an index set, and for all $j \in J$, let

$$A_j = \{(x, y, t) : (x, y) \in A_\theta, a_j(x, y) < t < b_j(x, y)\},$$

for some analytic, subanalytic functions $1 \le a_j < b_j$, where we also allow the case when $b_j \equiv +\infty$ on A_θ.

Suppose also that a_j, b_j, and $b_j - a_j$ are ψ-prepared on A_θ as follows:

$$\begin{aligned} a_j(x, y) &= a_{j,0}(x) y^{\alpha_j} u_{a_j}(x, y), \\ b_j(x, y) &= b_{j,0}(x) y^{\beta_j} u_{b_j}(x, y), \\ b_j(x, y) - a_j(x, y) &= c_{j,0}(x) y^{\delta_j} u_{c_j}(x, y) \end{aligned}$$

for some analytic, subanalytic functions $a_{j,0}, b_{j,0}, c_{j,0}$, some $\alpha_j, \beta_j, \delta_j \in \mathbb{Q}$, and some ψ-units $u_{a_j}, u_{b_j}, u_{c_j}$. (When $b_j = +\infty$ we stipulate that $b_{j,0} = c_{j,0} = +\infty$, $\beta_j = \delta_j = 0$, and $u_{b_j} = u_{c_j} = 1$.)

In this situation, given $d_j \in \mathbb{N} \setminus \{0\}$, we define the bounded, analytic, subanalytic map ψ_j on A_j as

$$\psi_j(x, y, t) = \Big(\psi(x, y), \Big(\frac{a_{j,0}(x) y^{\alpha_j}}{t}\Big)^{1/d_j}, \Big(\frac{t}{b_{j,0}(x) y^{\beta_j}}\Big)^{1/d_j}\Big) \quad \text{if } b_j < +\infty, \tag{25}$$

and $$\psi_j(x, y, t) = \Big(\psi(x, y), \Big(\frac{a_{j,0}(x) y^{\alpha_j}}{t}\Big)^{1/d_j}\Big) \quad \text{if } b_j \equiv +\infty.$$

The next proposition establishes that, after writing f as a sum of generators and after preparing suitably all the subanalytic and constructible functions appearing in the generators, we obtain a decomposition of $X \times \mathbb{R}$ into cells over which each of the generators has a well organized form. In particular, the generators are superintegrable over every cell in the partition whose fibers over $\mathbb{R}^m$ are bounded (see Remark 5.7(2) below).

PROPOSITION 5.5

Let $f \in \mathcal{C}^{\exp}(X \times \mathbb{R})$, for some subanalytic set $X \subseteq \mathbb{R}^m$. Then there exists a finite partition $\mathcal{A}$ of $X \times \mathbb{R}$ into cells over $\mathbb{R}^m$ such that, for each $A \in \mathcal{A}$ that is open over $\mathbb{R}^m$, there exists a center θ for A for which we may express $f \circ P_\theta$ as a finite sum

$$f \circ P_\theta(x, y) = \sum_{j \in J} T_j(x, y) \tag{26}$$

on $A_\theta = \{(x, y) : x \in \Pi_m(A), a(x) < y < b(x)\}$, where each T_j is a generator for $\mathcal{C}^{\exp}(A_\theta)$ of the form

$$T_j(x,y) = f_j(x) y^{p_j} (\log y)^{q_j} \mathrm{e}^{\mathrm{i}\phi_j(x,y)} \gamma_j(x,y) \tag{27}$$

for some $f_j \in \mathcal{C}(\Pi_m(A))$, $p_j \in \mathbb{Q}$, $q_j \in \mathbb{N}$, $\phi_j \in \mathcal{S}(A_\theta)$, *and function* γ_j, *where*

$$\gamma_j(x,y) = \int_{a_j(x,y)}^{b_j(x,y)} \Gamma_j(x,y,t)\,\mathrm{d}t \tag{28}$$

with

$$\Gamma_j(x,y,t) = t^{r_j} h_j(x,y,t)(\log t)^{s_j} \mathrm{e}^{\mathrm{i}\sigma_j t}$$

for some $r_j \in \mathbb{Q}$, $s_j \in \mathbb{N}$, $\sigma_j \in \{-1,1\}$, *analytic, subanalytic functions* a_j, b_j *as in Definition 5.4, and some* ψ_j*-function* h_j *(where* ψ_j *is as in (25), for some* d, ψ, d_j*). We may furthermore assume that the rational numbers* α_j, β_j, *and* δ_j *(see Definition 5.4) are integer multiples of* $1/d$.

Proof
Write f as a finite sum of generators for $\mathcal{C}^{\exp}(X \times \mathbb{R})$, say,

$$f(x,y) = \sum_{j \in J} T_j(x,y),$$

where

$$T_j(x,y) = g_j(x,y) \mathrm{e}^{\mathrm{i}\phi_j(x,y)} \gamma_j(x,y),$$

with

$$\gamma_j(x,y) = \int_{\mathbb{R}} H_j(x,y,t) \big(\log|t|\big)^{\ell_j} \mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t.$$

Apply Proposition 3.10 (in the form in Remark 3.11) to the collection

$$\big\{H_j(x,y,t)\big(\log|t|\big)^{\ell_j}\big\}_{j\in J} \subseteq \mathcal{C}(X \times \mathbb{R} \times \mathbb{R}). \tag{29}$$

This gives a finite partition $\mathcal{B}$ of $(X \times \mathbb{R}) \times \mathbb{R}$ into cells over $\mathbb{R}^{m+1}$. By further partitioning in (x,y), we may assume that $\mathcal{A} := \{\Pi_{m+1}(B) : B \in \mathcal{B}\}$ is a partition of $X \times \mathbb{R}$. By working piecewise, we may focus on one $A \in \mathcal{A}$. There are finitely many disjoint cells $B \in \mathcal{B}$ such that $\Pi_{m+1}(B) = A$. Pick one such B which is open over $\mathbb{R}^{m+1}$. Write

$$B = \big\{(x,y,t) : (x,y) \in A, \widetilde{a}(x,y) < t - \Theta(x,y) < \widetilde{b}(x,y)\big\},$$

where Θ is the center given by the preparation of the collection in (29). We fix an element of this collection, and we focus on one summand of the preparation of such an element. This will have the form

$$f_0(x,y)\big|t-\Theta(x,y)\big|^r\big(\log\big|y-\Theta(x,y)\big|\big)^s h(x,y,t),$$

where $f_0 \in \mathcal{C}(A)$ and h is a Ψ-function (for a suitable bounded subanalytic Ψ).

We write $\mathrm{e}^{\mathrm{i}t} = \mathrm{e}^{\mathrm{i}(t-\Theta(x,y))}\mathrm{e}^{\mathrm{i}\Theta(x,y)}$. By factoring out of the integral the term $f_0(x,y)\mathrm{e}^{\mathrm{i}\Theta(x,y)}$ and by absorbing f_0 in the constructible coefficient g and $\mathrm{e}^{\mathrm{i}\Theta(x,y)}$ in the exponential term $\mathrm{e}^{\mathrm{i}\phi(x,y)}$, we can reduce to studying generators of the form

$$g(x,y)\mathrm{e}^{\mathrm{i}\phi(x,y)}\int_{\widetilde{a}(x,y)}^{\widetilde{b}(x,y)}\big|t-\Theta(x,y)\big|^r h(x,y,t)\big(\log\big|t-\Theta(x,y)\big|\big)^s\mathrm{e}^{\mathrm{i}(t-\Theta(x,y))}\,\mathrm{d}t \tag{30}$$

on A. Now, the set

$$\big\{t-\Theta(x,y) : (x,y,t)\in B\big\} \tag{31}$$

is contained in one of the sets $(-\infty,-1)$, $(-1,0)$, $(0,1)$, or $(1,+\infty)$.

Suppose first that (31) is contained in either $(-1,0)$ or $(0,1)$. Then $\mathrm{e}^{\mathrm{i}(t-\Theta(x,y))}$ is a complex-valued subanalytic function on A (see Definition 2.3), so the integral in (30) is a complex-valued constructible function on A. This implies that (30) is in $\mathcal{C}^{\mathrm{exp}}_{\mathrm{naive}}(A)$ (because $\mathcal{C}$ is stable under integration); hence, we can apply Proposition 3.10 to the constructible part of (30), preparing it with respect to the variable y. Now, we can view the ψ-function obtained in this preparation as a γ-function of the form (28) (see Remark 2.6), and we are done.

Now suppose that (31) is contained in $(-\infty,-1)$ or $(1,+\infty)$. Then by applying the change of coordinates $t\mapsto \sigma t+\Theta(x,y)$ for an appropriate choice of $\sigma\in\{-1,1\}$ and adjusting the definitions of $\widetilde{a}$, $\widetilde{b}$, and h accordingly, we may assume that $1\le \widetilde{a}(x,y)<\widetilde{b}(x,y)$ and that (30) is of the form

$$g(x,y)\mathrm{e}^{\mathrm{i}\phi(x,y)}\int_{\widetilde{a}(x,y)}^{\widetilde{b}(x,y)} t^r h(x,y,t)(\log t)^s \mathrm{e}^{\mathrm{i}\sigma t}\,\mathrm{d}t.$$

Summing up, we have constructed a finite partition $\mathcal{A}$ of $X\times\mathbb{R}$ into subanalytic sets such that for each $A\in\mathcal{A}$ we may write f as a finite sum

$$f(x,y)=\sum_{j\in J} g_j(x,y)\mathrm{e}^{\mathrm{i}\phi_j(x,y)}\gamma_j(x,y) \tag{32}$$

on A, where $g_j\in\mathcal{C}(A)$, $\phi_j\in\mathcal{S}(A)$, and

$$\gamma_j(x,y)=\int_{\widetilde{a}_j(x,y)}^{\widetilde{b}_j(x,y)}\widetilde{\Gamma}_j(x,y,t)\,\mathrm{d}t \tag{33}$$

with

$$\widetilde{\Gamma}_j(x,y,t)=t^{r_j}\widetilde{h}_j(x,y,t)(\log t)^{s_j}\mathrm{e}^{\mathrm{i}\sigma_j t}, \tag{34}$$

where $1\le \widetilde{a}_j<\widetilde{b}_j$ (with either $\widetilde{b}_j<+\infty$ or $\widetilde{b}_j\equiv+\infty$), $r_j\in\mathbb{Q}$, $s_j\in\mathbb{N}$, $\sigma_j\in\{-1,1\}$, and $\widetilde{h}_j$ is a $\widetilde{\psi}_j$-function, with

$$\widetilde{\psi}_j(x,y,t)=\Big(\widetilde{c}_{j,1}(x,y),\dots,\widetilde{c}_{j,N_j}(x,y),\Big(\frac{\widetilde{a}_j(x,y)}{t}\Big)^{1/d_j},\Big(\frac{t}{\widetilde{b}_j(x,y)}\Big)^{1/d_j}\Big)$$
$$\text{if } \widetilde{b}_j<\infty,$$

and $$\widetilde{\psi}_j(x,y,t)=\Big(\widetilde{c}_{j,1}(x,y),\dots,\widetilde{c}_{j,N_j}(x,y),\Big(\frac{\widetilde{a}_j(x,y)}{t}\Big)^{1/d_j}\Big)\quad\text{if } \widetilde{b}_j\equiv+\infty,$$

defined on

$$\widetilde{A}_j=\big\{(x,y,t):(x,y)\in A,\widetilde{a}_j(x,y)<t<\widetilde{b}_j(x,y)\big\}.$$

We may additionally assume that the positive integer d_j has been chosen so that r_j is an integer multiple of $1/d_j$.

In order to have a more uniform notation, we will assume that $\widetilde{\psi}_j$ maps into $\mathbb{R}^{N_j+2}$ for each $j\in J$. (This is the case when $\widetilde{b}_j<+\infty$, and the argument adapts to the case in which $\widetilde{b}_j\equiv+\infty$ by simply ignoring the last component of $\widetilde{\psi}_j$ involving $(\frac{t}{\widetilde{b}_j(x,y)})^{1/d_j}$.) For each $j\in J$, fix $p_j=(p_{j,1},\dots,p_{j,N_j+2})$ and $\eta_j=(\eta_{j,1},\dots,\eta_{j,N_j+2})$ in $(0,\infty)^{N_j+2}$ and also a real analytic function $\widetilde{H}_j$ on $B_p(\mathbb{R})$ such that $\widetilde{\psi}_j(\widetilde{A}_j)\subseteq B_p(\mathbb{R})$, $\widetilde{h}_j=\widetilde{H}_j\circ\widetilde{\psi}_j$, and $\widetilde{H}_j$ extends to a complex analytic function on a neighborhood of $B_{p+\eta}(\mathbb{C})$. We may assume that $p_{j,N_j+1}=p_{j,N_j+2}=1$. Fix $\varepsilon\in(0,1)$ sufficiently small so that, for all $j\in J$ and $k\in\{1,\dots,N_j+2\}$,

$$\begin{cases}\frac{1+\varepsilon}{1-\varepsilon}p_{j,k}<p_{j,k}+\eta_{j,k} & \text{if } k\in\{1,\dots,N_j\},\\ (\frac{1+\varepsilon}{1-\varepsilon})^{1/d_j}<1+\eta_{j,k} & \text{if } k=N_j+1 \text{ or } k=N_j+2.\end{cases} \tag{35}$$

For each set $A\in\mathcal{A}$, apply Proposition 3.10 (with respect to the variable y) to

$$\{g_j\}_{j\in J}\subseteq\mathcal{C}(A)\qquad\text{and}\qquad\{\widetilde{c}_{j,1},\dots,\widetilde{c}_{j,N_j},\widetilde{a}_j,\widetilde{b}_j,\widetilde{b}_j-\widetilde{a}_j\}_{j\in J}\subseteq\mathcal{S}(A)$$

so that the units occurring in the preparation of $\{\widetilde{c}_{j,1},\dots,\widetilde{c}_{j,N_j},\widetilde{a}_j,\widetilde{b}_j,\widetilde{b}_j-\widetilde{a}_j\}_{j\in J}$ are within ε of 1 (see Remark 3.12), and then redefine $\mathcal{A}$ to be the finer partition of $X\times\mathbb{R}$ into the cells over $\mathbb{R}^m$ thus created.

Focus on one cell $A\in\mathcal{A}$ that is open over $\mathbb{R}^m$, and let θ be the center of A given by the preparation. We now use the notation set up in Definition 5.4, where $a_j=\widetilde{a}_j\circ P_\theta$ and $b_j=\widetilde{b}_j\circ P_\theta$ and where the positive integer d in Definition 5.4 has been chosen to be a common denominator of the set of rational exponents $\{\alpha_j,\beta_j,\delta_j : j\in J\}$ and also of the rational exponents of y in the ψ-prepared forms of each of the functions $c_{j,k} := \widetilde{c}_{j,k} \circ P_\theta$ with $j \in J$ and $k \in \{1, \dots, N_j\}$. Since each constructible function $g_j \circ P_\theta$ is prepared on A_θ, it is apparent from (32), (33), and (34) that $f \circ P_\theta$ is of the form asserted in the conclusion of the proposition except for one detail: although each function $h_j(x, y, t) := \widetilde{h}_j(P_\theta(x, y), t)$ is clearly a $\widetilde{\psi}_j(P_\theta(x, y), t)$-function, the conclusion of the proposition asserts that h_j is a ψ_j-function for the map ψ_j defined in Definition 5.4. To finish the proof, we will show that h_j is a ψ_j-function after ψ_j is modified by extending its list of component functions $c_1(x), \dots, c_N(x)$ in x alone by some additional functions in x obtained from the ψ-prepared forms of the functions in $\{c_{j,k}\}_{j,k}$.

In order to have a more uniform notation when showing this, we will assume that ψ maps into $\mathbb{R}^{N+2}$ (as would be the case when $b < +\infty$). For each $j \in J$, define K_j^+ to be the set of all $k \in \{1, \dots, N_j\}$ such that the exponent of y in the ψ-prepared form of $c_{j,k}$ is greater than or equal to 0, and define $K_j^- = \{1, \dots, N_j\} \setminus K_j^+$. For each $j \in J$ and $k \in \{1, \dots, N_j\}$, we may write

$$c_{j,k}(x, y) = \begin{cases} c_{j,k,0}(x)(\frac{y}{b(x)})^{\nu_{j,k}/d} v_{j,k}(x, y) & \text{if } k \in K_j^+, \\ c_{j,k,0}(x)(\frac{a(x)}{y})^{\nu_{j,k}/d} v_{j,k}(x, y) & \text{if } k \in K_j^-, \end{cases}$$

for some $c_{j,k,0} \in \mathcal{S}(\Pi_m(A))$, $\nu_{j,k} \in \mathbb{N}$, and ψ-unit $v_{j,k}$. Fix $q = (q_1, \dots, q_{N+2})$ in $(0, \infty)^{N+2}$ such that $\psi(A_\theta) \subseteq B_q(\mathbb{R})$ and such that for all $j \in J$ and $k \in \{1, \dots, N_j\}$ we have $u_{a_j} = U_{a_j} \circ \psi$, $u_{b_j} = U_{b_j} \circ \psi$, and $v_{j,k} = V_{j,k} \circ \psi$ for some real analytic functions U_{a_j}, U_{b_j}, and $V_{j,k}$ on $B_q(\mathbb{R})$ which extend to complex analytic functions on a neighborhood of $B_q(\mathbb{C})$ such that, for each $U \in \{U_{a_j}, U_{b_j}, V_{j,k}\}_{j,k}$,

$$\big|U(z) - 1\big| < \varepsilon \quad \text{for all } z \in B_q(\mathbb{C}).$$

We may assume that $q_{N+1} = q_{N+2} = 1$.

Focus on one choice of $j \in J$. Writing out the equation $h_j(x, y, t) = \widetilde{H}_j \circ \widetilde{\psi}_j(P_\theta(x, y), t)$ in full detail with the ψ-prepared forms of its components gives

$$\begin{aligned} h_j(x, y, t) = \widetilde{H}_j\bigg(&\Big(c_{j,k,0}(x)\Big(\frac{a(x)}{y}\Big)^{\nu_{j,k}/d} V_{j,k} \circ \psi(x, y)\Big)_{k \in K_j^-}, \\ &\Big(c_{j,k,0}(x)\Big(\frac{y}{b(x)}\Big)^{\nu_{j,k}/d} V_{j,k} \circ \psi(x, y)\Big)_{k \in K_j^+}, \\ &\Big(\frac{a_{j,0}(x) y^{\alpha_j}}{t}\Big)^{1/d_j} \big(U_{a_j} \circ \psi(x, y)\big)^{1/d_j}, \\ &\Big(\frac{t}{b_{j,0}(x) y^{\beta_j}}\Big)^{1/d_j} \big(U_{b_j} \circ \psi(x, y)\big)^{-1/d_j}\bigg). \end{aligned} \tag{36}$$

Consider $k \in \{1, \dots, N_j\}$, and observe that on A_θ we have that $|c_{j,k}(x, y)| \le p_{j,k}$, that $|v_{j,k}(x, y)| \ge 1 - \varepsilon$, and that $\frac{a(x)}{y}$ and $\frac{y}{b(x)}$ can take values arbitrarily close to 1 (for each fixed $x \in \Pi_m(A)$). It follows that

$$\left|c_{j,k,0}(x)\right| \le \frac{p_{j,k}}{1-\varepsilon} \tag{37}$$

on A_θ. Similar reasoning shows that

$$\left|\frac{a_{j,0}(x)y^{\alpha_j}}{t}\right|^{1/d_j} \le \Big(\frac{1}{1-\varepsilon}\Big)^{1/d_j} \qquad \text{and} \qquad \left|\frac{t}{b_{j,0}(x)y^{\beta_j}}\right|^{1/d_j} \le (1+\varepsilon)^{1/d_j} \tag{38}$$

hold on A_j as well. Clearly

$$\left|\frac{a(x)}{y}\right|^{1/d} \le 1 \qquad \text{and} \qquad \left|\frac{y}{b(x)}\right|^{1/d} \le 1 \tag{39}$$

on A_θ, and also for all $k \in \{1, \dots, N_j\}$ we have

$$\begin{aligned} |V_{j,k}| \le 1+\varepsilon, \qquad |U_{a_j}|^{1/d_j} \le (1+\varepsilon)^{1/d_j}, \qquad \text{and} \\ |U_{b_j}|^{-1/d_j} \le \Big(\frac{1}{1-\varepsilon}\Big)^{1/d_j} \end{aligned} \tag{40}$$

on $B_q(\mathbb{C})$. Using the variables $(W, X, Y, Z) = ((W_k)_{k=1}^N, (X_{j,k})_{k=1}^{N_j}, Y_1, Y_2, Z_1, Z_2)$, define

$$\begin{aligned} H_j(W, X, Y, Z) := \widetilde{H}_j\big(\big(X_k Y_1^{\nu_{j,k}} V_{j,k}(W)\big)_{k \in K_j^-}, \big(X_k Y_2^{\nu_{j,k}} V_{j,k}(W)\big)_{k \in K_j^+}, \\ Z_1\big(U_{a_j}(W)\big)^{1/d_j}, Z_2\big(U_{b_j}(W)\big)^{-1/d_j}\big). \end{aligned}$$

Define

$$\rho = \Big(q_1, \dots, q_N, \frac{p_1}{1-\varepsilon}, \dots, \frac{p_{N_j}}{1-\varepsilon}, 1, 1, 1, 1\Big).$$

Observe from the inequalities (37)–(40), from the conditions (35) imposed upon our choice of ε, and from (36) that the range of the map on A_j given by

$$\begin{aligned} (x, y, t) \mapsto \Big(\big(c_k(x)\big)_{k=1}^N, \big(c_{j,k}(x)\big)_{k=1}^{N_j}, \Big(\frac{a(x)}{y}\Big)^{1/d}, \Big(\frac{y}{b(x)}\Big)^{1/d}, \\ \Big(\frac{a_{j,0}(x)y^{\alpha_j}}{t}\Big)^{1/d_j}, \Big(\frac{t}{b_{j,0}(x)y^{\beta_j}}\Big)^{1/d_j}\Big) \end{aligned} \tag{41}$$

is contained in $B_\rho(\mathbb{R})$, that H_j is defined as a complex analytic function on a neighborhood of $B_\rho(\mathbb{C})$, and that h_j is the composition of H_j with the map (41). This completes the proof. □

Definition 5.6
We call a generator for $\mathcal{C}^{\exp}(A_\theta)$ of the form (27) a *prepared generator*.

Remarks 5.7
Fix a prepared generator T_j as in Proposition 5.5.

(1) If $b_j < +\infty$, then we may suppose that $r_j = 0$. If $b_j \equiv +\infty$, then we may suppose that $r_j < -1$.
To see this, suppose first that $b_j < +\infty$. If $r_j \geq 0$, then write

$$\begin{aligned} &g_j(x,y)t^{r_j}h_j(x,y,t) \\ &\quad = \big(g_j(x,y)\big(b_{j,0}(x)y^{\beta_j}\big)^{r_j}\big)\Big(\Big(\frac{t}{b_{j,0}(x)y^{\beta_j}}\Big)^{r_j}h_j(x,y,t)\Big) \\ &\quad = \widetilde{g_j}(x,y)\widetilde{h}_j(x,y,t). \end{aligned}$$

If $r_j < 0$, then write

$$\begin{aligned} &g_j(x,y)t^{r_j}h_j(x,y,t) \\ &\quad = \big(g_j(x,y)\big(a_{j,0}(x)y^{\alpha_j}\big)^{r_j}\big)\Big(\Big(\frac{a_{j,0}(x)y^{\alpha_j}}{t}\Big)^{-r_j}h_j(x,y,t)\Big) \\ &\quad = \widetilde{g_j}(x,y)\widetilde{h}_j(x,y,t). \end{aligned}$$

Note that, in both cases, $\widetilde{h}_j$ is a ψ_j-function (but not necessarily a ψ_j-unit), because r_j is an integral multiple of $1/d_j$. We have hence reduced to the case $r_j = 0$. Suppose now that $b_j \equiv +\infty$. Let n_0 be the smallest exponent appearing in the series expansion of $h_j(x,y,t)$ with respect to the variable $(\frac{a_{j,0}(x)y^{\alpha_j}}{t})^{1/d_j}$. Then we can factor out the power $(\frac{a_{j,0}(x)y^{\alpha_j}}{t})^{n_0/d_j}$ from the expansion of h_j and write

$$g_j(x,y)t^{r_j}h_j(x,y,t) = \big(g_j(x,y)\big(a_{j,0}(x)y^{\alpha_j}\big)^{n_0/d_j}\big)t^{r_j-n_0/d_j}\widetilde{h}_j(x,y,t),$$

where $\widetilde{h}_j$ is a ψ_j-unit. Note that $\widetilde{r_j} := r_j - n_0/d_j$ is necessarily strictly smaller than -1. (Otherwise γ_j would not be defined.)

(2) Whenever $b < +\infty$, T_j is superintegrable over $\Pi_m(A)$. This is clear, since for all $x \in \Pi_m(A)$, $y \mapsto T_j^{\mathrm{abs}}(x,y)$ extends to a continuous function on the closure of A_x in $\mathbb{R}$.

Remark 5.8
For all $m \in \mathbb{N}$, subanalytic $X \subseteq \mathbb{R}^m$, and $g \in \mathcal{C}^{\exp}(X)$, there exists a finite partition

$\mathcal{A}$ of X into subanalytic cells (see Definition 3.3) such that $g \restriction A$ is analytic for each open set $A \in \mathcal{A}$.

Proof

Apply Proposition 5.5 to f (except we now omit the variable y since we are working on X rather than on $X \times \mathbb{R}$), and let $\mathcal{A}$ be the partition of X so obtained. Consider an open set $A \in \mathcal{A}$. For each $j \in J$, $\gamma_j \restriction A$ is analytic because it is the integral of an analytic function with analytic limits of integration. (Namely, basic facts about power series show that the antiderivative in t of the integrand $\Gamma_j(x,t)$ is analytic, and evaluating this antiderivative at analytic limits of integration in x gives an analytic function in x.) It therefore follows from (26) and (27) that $f \restriction A$ is analytic. $\square$

In view of Remark 5.7(2), we can focus our attention on cells which are unbounded above. For such cells, our next goal is to reduce to the case where each of the generators in (26) is either superintegrable or in $\mathcal{C}^{\exp}_{\text{naive}}(A_\theta)$, or is such that the variable y does not appear in the integration limits a_j and b_j of the γ-function.

PROPOSITION 5.9

Proposition 5.5 holds with the additional property that, whenever $b \equiv +\infty$, every T_j is either superintegrable or in $\mathcal{C}^{\exp}_{\text{naive}}(A_\theta)$, or there exist analytic and subanalytic functions $a_{j,0}, b_{j,0}$ on $\Pi_m(A)$ such that $a_j(x,y) = a_{j,0}(x)$ and either $b_j \equiv +\infty$ or $b_j(x,y) = b_{j,0}(x)$.

In order to prove the above proposition, we first need to establish two technical lemmas (Lemmas 5.12 and 5.13 below). Their aim is to reduce to the case of prepared generators such that the variable y only appears in the units in the prepared form of the integration limits of the γ-function; that is, $\alpha_j = \beta_j = 0$. To achieve this, our main tool will be to compute γ by integrating by parts. This will lead to rewriting the prepared generator as a finite sum of generators which are either superintegrable or in $\mathcal{C}^{\exp}_{\text{naive}}(A_\theta)$, or are in a better form. (For example, the variable y now only appears in one of the two integration limits.) We will also have to refine the partition into cells along the way. This is harmless when we refine the partition with respect to the variables x. When further partitioning with respect to the variable y, we will possibly create new bounded cells, which can be handled as in Remark 5.7(2).

Definition 5.10

In the notation of Definition 5.4, suppose that $b_j < +\infty$. We let $\psi_{j,-}$ and $\psi_{j,+}$ be the maps obtained from ψ_j by omitting the last and the second-to-last components of

ψ_j, respectively. We extend this definition to the case $b_j \equiv +\infty$ by stipulating that $\psi_{j,-} = \psi_j$ and $\psi_{j,+} = 0$.

Remark 5.11
Note that, when $b_j \equiv +\infty$, α_j and β_j are necessarily nonnegative, since $a_j, b_j \geq 1$.

LEMMA 5.12 (Splitting)
Let $f \in \mathcal{C}^{\exp}(X)$ for some subanalytic set $X \subseteq \mathbb{R}^m$, and let $A \in \mathcal{A}$ be one of the cells obtained from Proposition 5.5 satisfying $b = +\infty$. Let T_j be one of the generators corresponding to this A satisfying $b_j < +\infty$, $\alpha_j = 0$, and $\beta_j > 0$. Then we may write T_j as a finite sum $\sum T_k$ of prepared generators, where each T_k is either superintegrable or in $\mathcal{C}^{\exp}_{\text{naive}}(A_\theta)$, or is such that $b_k \equiv +\infty$ (and hence h_k is a $\psi_{k,-}$-function).

Proof
We consider a generator T_j as in the statement of the lemma. In the notation of Proposition 5.5, let $n_j \in \mathbb{N}$ be such that

$$p_j - n_j \beta_j + \delta_j < -1. \tag{42}$$

Our next aim is to write h_j as a sum of three terms, depending on the choice of n_j, as follows:

$$\begin{aligned} h_j(x, y, t) = {} & \Big(\frac{t}{a_{j,0}(x)}\Big)^{r_{j,-}} h_{j,-}(x, y, t) + h_{j,0}(x, y, t) \\ & + \Big(\frac{t}{b_{j,0}(x) y^{\beta_j}}\Big)^{n_j} h_{j,+}(x, y, t), \end{aligned} \tag{43}$$

where $r_{j,-} < -1$ is rational, $h_{j,-}$ is a $\psi_{j,-}$-function, $h_{j,+}$ is a $\psi_{j,+}$-function, and $h_{j,0}$ is a finite sum of terms of the form $g(x, y) z^k$, where g is a ψ-function, $k \in \mathbb{N}$, and z is either $(\frac{t}{b_j})^{1/d_j}$ or $(\frac{a_j}{t})^{1/d_j}$.

In order to do this, we expand h_j as a series in the variables $(\frac{t}{b_j})^{1/d_j}, (\frac{a_j}{t})^{1/d_j}$, with ψ-functions as coefficients. Now, recalling that $b \equiv +\infty$ and $\alpha_j = 0$, for each $k, l \in \mathbb{N}$, write

$$\begin{aligned} & \Big(\frac{t}{b_{j,0}(x) y^{\beta_j}}\Big)^{k/d_j} \Big(\frac{a_{j,0}(x)}{t}\Big)^{l/d_j} \\ & = \begin{cases} (\frac{a_{j,0}(x)}{b_{j,0}(x) y^{\beta_j}})^{l/d_j} (\frac{t}{b_{j,0}(x) y^{\beta_j}})^{(k-l)/d_j} & \text{if } k \geq l, \\ (\frac{a_{j,0}(x)}{b_{j,0}(x) y^{\beta_j}})^{k/d_j} (\frac{a_{j,0}(x)}{t})^{(l-k)/d_j} & \text{if } k < l, \end{cases} \end{aligned}$$

and

$$\frac{a_{j,0}(x)}{b_{j,0}(x)y^{\beta_j}} = \Big[\frac{a_{j,0}(x)}{b_{j,0}(x)a(x)^{\beta_j}}\Big]\Big(\frac{a(x)}{y}\Big)^{\beta_j}. \tag{44}$$

The quotient on the left-hand side of (44) is bounded, since it is equal to the bounded quotient $\frac{a_j}{b_j}$ multiplied by a unit. Moreover, for each x we may take y to be arbitrarily close to $a(x)$, thereby making $\frac{a(x)}{y}$ arbitrarily close to 1. It follows that the function in square brackets on the right-hand side of (44), which does not depend on y, is also bounded and therefore can be included in the list of functions $c_1(x), \dots, c_N(x)$ in ψ.

Therefore, we can write h_j as the sum of a $\psi_{j,-}$-function of the form $\sum_{k\geq 1} g_{j,k}(x,y)(\frac{a_{j,0}(x)}{t})^{k/d_j}$ plus a $\psi_{j,+}$-function of the form $\sum_{k\geq 0} \widetilde{g}_{j,k}(x, y)(\frac{t}{b_{j,0}(x)y^{\beta_j}})^{k/d_j}$, where $g_{j,k}, \widetilde{g}_{j,k}$ are ψ-functions. If we set

$$h_{j,-} = \sum_{k\geq d_j+1} g_{j,k}(x,y)\Big(\frac{a_{j,0}(x)}{t}\Big)^{\frac{k}{d_j}-1-\frac{1}{d_j}},$$

$$h_{j,+} = \sum_{k\geq d_j n_j} \widetilde{g}_{j,k}(x,y)\Big(\frac{t}{b_{j,0}(x)y^{\beta_j}}\Big)^{\frac{k}{d_j}-n_j},$$

then we obtain (43) with $r_{j,-} = -1 - \frac{1}{d_j}$ and

$$h_{j,0} = \sum_{k=1}^{d_j} g_{j,k}(x,y)\Big(\frac{a_{j,0}(x)}{t}\Big)^{k/d_j} + \sum_{k=0}^{d_j n_j - 1} \widetilde{g}_{j,k}(x,y)\Big(\frac{t}{b_{j,0}(x)y^{\beta_j}}\Big)^{k/d_j}.$$

Hence, we can write

$$\begin{aligned}
\Gamma_{j,-}(x,y,t) &= \Big(\frac{t}{a_{j,0}(x)}\Big)^{r_{j,-}} h_{j,-}(x,y,t)(\log t)^{s_j}\,\mathrm{e}^{\mathrm{i}\sigma_j t},\\
\Gamma_{j,0}(x,y,t) &= h_{j,0}(x,y,t)(\log t)^{s_j}\,\mathrm{e}^{\mathrm{i}\sigma_j t},\\
\Gamma_{j,+}(x,y,t) &= \Big(\frac{t}{b_{j,0}(x)y^{\beta_j}}\Big)^{n_j} h_{j,+}(x,y,t)(\log t)^{s_j}\,\mathrm{e}^{\mathrm{i}\sigma_j t},
\end{aligned}$$

and

$$T_j(x,y) = T_{j,-}(x,y) + T_{j,0}(x,y) + T_{j,+}(x,y),$$

where one obtains $T_{j,-}$, $T_{j,0}$, and $T_{j,+}$ from T_j by replacing Γ_j with $\Gamma_{j,-}$, $\Gamma_{j,0}$, and $\Gamma_{j,+}$, respectively.

To handle $T_{j,-}$, note that, since $r_{j,-} < -1$, we can use the additivity relation

$$\int_{a_j(x,y)}^{b_j(x,y)} \Gamma_{j,-}(x,y,t)\,\mathrm{d}t = \int_{a_j(x,y)}^{+\infty} \Gamma_{j,-}(x,y,t)\,\mathrm{d}t - \int_{b_j(x,y)}^{+\infty} \Gamma_{j,-}(x,y,t)\,\mathrm{d}t.$$

Therefore, we can replace $T_{j,-}$ by a sum of two prepared generators for $\mathcal{C}^{\exp}(A_\theta)$ whose γ-functions are defined by integrals with $+\infty$ as the upper limit of integration.

To handle $T_{j,0}$, compute $\int_{a_j(x,y)}^{b_j(x,y)} \Gamma_{j,0}(x,y,t)\,\mathrm{d}t$ by integrating by parts, where one differentiates $h_{j,0}(x,y,t)(\log t)^{s_j}$ and integrates $\mathrm{e}^{\mathrm{i}\sigma_j t}$. This has the effect of replacing $T_{j,0}(x,y)$ with a sum of terms that are either prepared generators for $\mathcal{C}^{\exp}_{\mathrm{naive}}(A_\theta)$ or are of the same form as $T_{j,0}$ but with the powers of t in $h_{j,0}$ reduced by 1. By repeating this strategy finitely many times, we reduce to the case in which all powers of t in $h_{j,0}$ are less than -1, which can then be handled as we did for $T_{j,-}$.

It remains to handle $T_{j,+}$. Recall that

$$\begin{aligned}\Big(\frac{t}{b_{j,0}(x)y^{\beta_j}}\Big)^{n_j} & h_{j,+}(x,y,t)(\log t)^{s_j} \\ &= \Big(\sum_{k=n_jd_j}^{+\infty} \widetilde{g}_{j,k}(x,y)\Big(\frac{t}{b_{j,0}(x)y^{\beta_j}}\Big)^{k/d_j}\Big)(\log t)^{s_j}. \end{aligned} \tag{45}$$

Differentiating the right-hand side of (45) with respect to t gives

$$\begin{aligned}\frac{1}{b_{j,0}(x)y^{\beta_j}}\Big(&\Big(\sum_{k=n_jd_j}^{+\infty} \frac{k}{d_j}\widetilde{g}_{j,k}(x,y)\Big(\frac{t}{b_{j,0}(x)y^{\beta_j}}\Big)^{k/d_j-1}\Big)(\log t)^{s_j} \\ &+ s_j\Big(\sum_{k=n_jd_j}^{+\infty} \widetilde{g}_{j,k}(x,y)\Big(\frac{t}{b_{j,0}(x)y^{\beta_j}}\Big)^{k/d_j-1}\Big)(\log t)^{s_j-1}\Big).\end{aligned}$$

Therefore, if we compute $\int_{a_j(x,y)}^{b_j(x,y)} \Gamma_{j,+}(x,y,t)\,\mathrm{d}t$ by integrating by parts n_j times, where we begin by differentiating the left-hand side of (45) and integrating $\mathrm{e}^{\mathrm{i}\sigma_j t}$ as before, then we reduce to studying prepared generators for $\mathcal{C}^{\exp}(A_\theta)$ of the form

$$T(x,y) = f_j(x)y^{p_j-n_j\beta_j}(\log y)^{q_j}\mathrm{e}^{\mathrm{i}\phi_j(x,y)}\int_{a_j(x,y)}^{b_j(x,y)} h(x,y,t)(\log t)^s\mathrm{e}^{\mathrm{i}\sigma_j t}\,\mathrm{d}t,$$

where h is a $\psi_{j,+}$-function and s is a rational number. Since h is bounded and the length of the interval $(a_j(x,y), b_j(x,y))$ is of order y^{δ_j} as $y \to +\infty$ (see Definition 5.4 and Proposition 5.5), it follows that for each $x \in \Pi_m(A)$ there is a constant $C(x) > 0$ such that

$$T^{\mathrm{abs}}(x,y) \le C(x)y^{p_j-n_j\beta_j+\delta_j}(\log y)^{q_j+s}.$$

Hence, by (42) we can conclude that T is superintegrable. □

LEMMA 5.13

In the notation of Proposition 5.5, suppose that $b \equiv +\infty$. If T_j is a prepared *generator with the property that $\alpha_j > 0$, then we may write T_j as a finite sum of prepared generators which are either superintegrable or in $\mathcal{C}^{\exp}_{\text{naive}}(A_\theta)$.*

Proof
We consider a generator T_j as in the statement of the lemma. In the notation of Proposition 5.5, suppose first that $b_j < +\infty$. By Remark 5.7(1), we have $r_j = 0$.

Assume that

$$p_j + \delta_j < -1.$$

Since h_j is bounded by a constant and the length of the interval $(a_j(x,y), b_j(x,y))$ is of order y^{δ_j} as $y \to +\infty$, it follows that for each $x \in \Pi_m(A)$ there is a constant $C(x) > 0$ such that

$$T_j^{\text{abs}}(x,y) \le C(x) y^{p_j+\delta_j} (\log y)^{q_j+s_j}. \tag{46}$$

So T_j is superintegrable, and we are done.

So now assume that $p_j + \delta_j \ge -1$. Note that

$$\frac{\partial}{\partial t}\Big(\Big(\frac{a_{j,0}(x)y^{\alpha_j}}{t}\Big)^{1/d_j}\Big) = -\frac{1}{d_j}\Big(\frac{a_{j,0}(x)y^{\alpha_j}}{t}\Big)^{1/d_j}\frac{1}{t}$$

and that

$$\frac{\partial}{\partial t}\Big(\Big(\frac{t}{b_{j,0}(x)y^{\beta_j}}\Big)^{1/d_j}\Big) = \frac{1}{d_j}\Big(\frac{t}{b_{j,0}(x)y^{\beta_j}}\Big)^{1/d_j}\frac{1}{t}.$$

Write $h_j = H_j \circ \psi_j$, where $H_j(X_1,\dots,X_N,Y,T_1,T_2)$ is a power series converging in a neighborhood of the closure of the image of ψ_j. Thus, we can factor out $1/t$ every time we differentiate the expression $h_j(x,y,t)(\log t)^{s_j}$ with respect to t. Moreover, the factor $1/t$ may be written as

$$\frac{1}{t} = \frac{1}{a_{j,0}(x)y^{\alpha_j}}\Big(\frac{a_{j,0}(x)y^{\alpha_j}}{t}\Big).$$

Therefore, if we compute $\gamma_j(x,y,t)$ by integrating by parts, where one integrates $e^{i\sigma_j t}$ and differentiates $h_j(x,y,t)(\log t)^{s_j}$, we can express T_j as a finite sum of terms, each of which is either in $\mathcal{C}^{\exp}_{\text{naive}}(A_\theta)$ or is of the same form as T_j, but with p_j replaced by $p_j - \alpha_j$. Therefore, by repeating this strategy finitely many times, we sufficiently decrease the value of p_j in order to reduce to the case in which $p_j + \delta_j < -1$, and we are done for the case $b_j < +\infty$.

It remains to consider the case $b_j \equiv +\infty$. By Remark 5.7(1), $r_j < -1$ and h_j is a $\psi_{j,-}$-function. This case can be handled very similarly to the previous case: one decreases the value of p_j by repeatedly integrating by parts in order to additionally assume that

$$p_j < -1.$$

Since

$$\gamma_j^{\mathrm{abs}}(x,y) \le M \int_1^{+\infty} t^{r_j} (\log t)^{s_j}\, \mathrm{d}t < +\infty,$$

where $|h_j| \le M$, this shows that the analogue of (46) is now

$$T_j^{\mathrm{abs}}(x,y) \le C(x) y^{p_j} (\log y)^{q_j}.$$

Hence, T_j is superintegrable. □

We now complete the proof of Proposition 5.9.

Proof of Proposition 5.9
Let $b \equiv +\infty$, and let T_j be as in Proposition 5.5. If T_j is either superintegrable or in $\mathcal{C}^{\exp}_{\mathrm{naive}}(A_\theta)$, then we are done. Otherwise, thanks to the lemmas above we may assume that $\alpha_j = \beta_j = 0$. (Recall that if $b_j \equiv +\infty$, we have set $b_{j,0} \equiv +\infty$ and $\beta_j = 0$.) To see this, if $\alpha_j > 0$, then apply Lemma 5.13. Suppose now that $\alpha_j = 0$. If $b_j \equiv +\infty$ or $\beta_j = 0$, then we are done. Otherwise, apply Lemma 5.12 and again Lemma 5.13.

We first establish the following claim: up to replacing $a(x)$ with some analytic subanalytic $\widetilde{a}(x) \ge a(x)$ and up to further partitioning with respect to the variables x, we may assume that, for all $j \in J$,

(1) $|a_j(x,y) - a_{j,0}(x)| \le 1$ and $|b_j(x,y) - b_{j,0}(x)| \le 1$ on A_θ, and

(2) the function h_j extends to a ψ_j-function with ψ_j now defined on the set

$$\begin{aligned}\widetilde{A_j} = \big\{&(x,y,t) : (x,y) \in A_\theta,\\ &\min\{a_{j,0}(x), a_j(x,y)\} < t < \max\{b_{j,0}(x), b_j(x,y)\}\big\}.\end{aligned}$$

To establish the claim, for each $j \in J$, fix a subanalytic neighborhood U_j of the closure of $\psi_j(A_j)$ such that $h_j = H_j \circ \psi_j$ for some power series H_j centered at the origin and converging on U_j. Recall that every ψ is bounded and subanalytic. Hence, for each ψ-unit $u \in \{u_{a_j}, u_{b_j}, u_{c_j}\}$, $\lim_{y\to+\infty} u(x,y)$ is a well-defined subanalytic function of x (which may be supposed to be analytic, up to refining the partition) and, therefore, may be considered as a part of the corresponding coefficient function in $\{a_{j,0}, b_{j,0}, c_{j,0}\}$. We may therefore assume that, for each $u \in \{u_{a_j}, u_{b_j}, u_{c_j}\}$,

$$\lim_{y\to+\infty} u(x,y) = 1.$$

In particular, $\lim_{y\to+\infty} a_j(x,y) = a_{j,0}(x)$ and $\lim_{y\to+\infty} b_j(x,y) = b_{j,0}(x)$. Hence, for each $x \in \Pi_m(A)$ there exists a real number $\widetilde{a}(x) \geq a(x)$ such that, for all $y > \widetilde{a}(x)$ and all $j \in J$, we have that $|a_j(x,y) - a_{j,0}(x)| \leq 1$, that $|b_j(x,y) - b_{j,0}(x)| \leq 1$, and that $\psi_j(\widetilde{A}_j) \subseteq U_j$. By definable choice (see, e.g., [30, Chapter 6]), we may take $\widetilde{a}$ to be a subanalytic function of x (and we may suppose $\widetilde{a}$ to be analytic, up to refining the partition). This establishes the claim.

We may therefore partition A_θ according to the conditions $a(x) < y < \widetilde{a}(x)$ and $\widetilde{a}(x) < y$. We are done on the subset of A_θ defined as $a(x) < y < \widetilde{a}(x)$ (as in the case of $b < +\infty$ treated in Proposition 5.5), so it suffices to consider the subset of A_θ defined by $y > \widetilde{a}(x)$. Therefore, up to changing notation, we may simply assume that $\widetilde{a}(x) = a(x)$.

Now, write

$$\int_{a_j(x,y)}^{b_j(x,y)} \Gamma_j(x,y,t)\,\mathrm{d}t = \int_{a_{j,0}(x)}^{b_{j,0}(x)} \Gamma_j(x,y,t)\,\mathrm{d}t - \int_{a_{j,0}(x)}^{a_j(x,y)} \Gamma_j(x,y,t)\,\mathrm{d}t + \int_{b_{j,0}(x)}^{b_j(x,y)} \Gamma_j(x,y,t)\,\mathrm{d}t.$$

(Note that when $b_j \equiv +\infty$ the last term of the sum does not appear.) We remark that

$$\int_{a_{j,0}(x)}^{a_j(x,y)} \Gamma_j(x,y,t)\,\mathrm{d}t = \mathrm{e}^{\mathrm{i}\sigma_j a_{j,0}(x)} \int_{a_{j,0}(x)}^{a_j(x,y)} t^{r_j} h_j(x,y,t)(\log t)^{s_j} \mathrm{e}^{\mathrm{i}\sigma_j(t-a_{j,0}(x))}\,\mathrm{d}t,$$

and thanks to the claim, $|a_j(x,y) - a_{j,0}(x)| \leq 1$ on A_θ. Hence, $\mathrm{e}^{\mathrm{i}\sigma_j(t-a_{j,0}(x))}$ is a complex-valued subanalytic function (see Definition 2.3) over its domain of integration. So the integral on the right-hand side of the above equation is a complex-valued constructible function on A_θ, because $\mathcal{C}$ is stable under integration (see [8], [9]). This shows that $(x,y) \mapsto \int_{a_{j,0}(x)}^{a_j(x,y)} \Gamma_j(x,y,t)\,\mathrm{d}t$ is in $\mathcal{C}^{\exp}_{\mathrm{naive}}(A_\theta)$. For similar reasons, $(x,y) \mapsto \int_{b_{j,0}(x)}^{b_j(x,y)} \Gamma_j(x,y,t)\,\mathrm{d}t$ is also in $\mathcal{C}^{\exp}_{\mathrm{naive}}(A_\theta)$. □

We are now ready to finish the proof of the preparation theorem. In view of Proposition 5.9, it only remains to show that those generators for which the variable y does not appear in the integration limits of the γ-function can be expressed as finite sums of generators which are either superintegrable or naive in y. Moreover, we need to ensure that Theorem 5.2(2b) is also satisfied.

Proof of Theorem 5.2

Let $b \equiv +\infty$, and consider a generator T_j as in the statement of Proposition 5.9, which is neither superintegrable nor in $\mathcal{C}^{\exp}_{\text{naive}}(A_\theta)$. Thus, T_j is such that $\gamma_j(x,y) = \int_{a_{j,0}(x)}^{b_{j,0}(x)} \Gamma_j(x,y,t)\,\mathrm{d}t$ (where we also allow the possibility $b_{j,0} \equiv +\infty$), and since $\alpha_j = \beta_j = 0$, the variable y now only appears in the component $(\frac{a(x)}{y})^{1/d}$ of ψ_j (see (24), (25)). Hence, we can now expand $h_j(x,y,t)$ as a power series in the variable $(a(x)/y)^{1/d}$ with coefficients in the variables (x,t). The powers of y which appear in T_j are thus of the form $p_j - \frac{n}{d}$, where n is the summation index in the power series expansion of h_j. Since finitely many of such powers are greater than or equal to -1, we can write T_j as a sum of finitely many terms that are naive in y plus a final term of the form

$$f_j(x) y^p (\log y)^{q_j} \mathrm{e}^{\mathrm{i}\phi_j(x,y)} \int_{a_{j,0}(x)}^{b_{j,0}(x)} t^{r_j} h(x,y,t)(\log t)^{s_j} \mathrm{e}^{\mathrm{i}\sigma_j t}\,\mathrm{d}t$$

for some rational $p < -1$ and ψ_j-function h. This final term is clearly superintegrable, since h is bounded. Summing up, we have written $f \circ P_\theta(x,y)$ as a finite sum of generators which are either superintegrable or of the form

$$T_j(x,y) = f_j(x) y^{r_j} (\log y)^{s_j} \mathrm{e}^{\mathrm{i}\phi_j(x,y)}, \tag{47}$$

where $f_j \in \mathcal{C}^{\exp}(\Pi_m(A))$, $r_j \in \mathbb{Q} \cap [-1,+\infty)$, $s_j \in \mathbb{N}$, and $\phi_j \in \mathcal{S}(A_\theta)$.

It remains to prove Theorem 5.2(2b). Let $J' = \{j : T_j \text{ is as in (47)}\}$, and apply Proposition 3.10 to the collection $\{\phi_j : j \in J'\}$. Focus on a cell $A' = \{(x,y) : x \in \Pi_m(A'), a'(x) < \sigma'(y - \theta'(x))^{\tau'} < b'(x)\} \subseteq A_\theta$ that this constructs, along with its associated center θ' and map ψ' given by

$$\psi'(x,y) = \Big(c'_1(x), \dots, c'_{N'}(x), \Big(\frac{a'(x)}{y}\Big)^{1/d'}, \Big(\frac{y}{b'(x)}\Big)^{1/d'}\Big) \quad \text{if } b' < +\infty,$$

and $$\psi'(x,y) = \Big(c'_1(x), \dots, c'_{N'}(x), \Big(\frac{a'(x)}{y}\Big)^{1/d'}\Big) \quad \text{if } b' \equiv +\infty$$

on

$$A'_{\theta'} = \big\{(x,y) : x \in \Pi_m(A'), a'(x) < y < b'(x)\big\}.$$

First suppose that $\theta' \neq 0$. Then the closure of $\{y/\theta'(x) : (x,y) \in A'\}$ is a compact subset of $(0,+\infty)$, so each of the fibers A'_x is bounded above. We are then done on A' by Remark 5.7(2).

Now suppose that $\theta' = 0$. Because $A'_x \subseteq (1,+\infty)$ for each x, it follows that $\sigma' = \tau' = 1$. Thus, $A' = \{(x,y) : x \in \Pi_m(A'), a'(x) < y < b'(x)\}$ with $1 \leq a \leq a' < b'$. When $b' < +\infty$, we are again done on A', so assume that $b' \equiv +\infty$. We may assume that the list of functions $c'_1, \dots, c'_{N'}$ contains $c_1, \dots, c_N$ and also $(a(x)/a'(x))^{1/d}$. We may also assume that d' was chosen so that $1/d$ is an integer multiple of $1/d'$. So because

$$\left(\frac{a(x)}{y}\right)^{1/d} = \left(\frac{a(x)}{a'(x)}\right)^{1/d} \left(\frac{a'(x)}{y}\right)^{1/d},$$

it follows that each component of ψ is a ψ'-function. Therefore, to simplify notation, we may simply assume that $A' = A_\theta$ and that $\psi' = \psi$.

Hence, on A_θ we can write, for all $j \in J'$,

$$\phi_j(x,y) = \phi_{j,0}(x) y^{l_j} u_j(x,y),$$

where $\phi_{j,0} \in \mathcal{S}(\Pi_m(A))$ is analytic, $l_j \in \mathbb{Q}$ (an integer multiple of $1/d$), and u_j is a ψ-unit. We expand the unit u_j with respect to the variable $(\frac{a(x)}{y})^{1/d}$ and multiply by y^{l_j}, so that we can rewrite the above equation as

$$\phi_j(x,y) = \phi_{j,+}(x,y) + \phi_{j,-}(x,y),$$

where $\phi_{j,+} \in \mathcal{S}(\Pi_m(A))[y^{1/d}]$ and $\phi_{j,-}(x,y) = \phi_{j,0}(x)(\frac{a(x)}{y})^{1/d}\widetilde{u}_j(x,y)$, for some ψ-unit $\widetilde{u}_j$. Up to refining the partition with respect to the variables x, we may assume that $|\phi_{j,0}|$ is either bounded from above or bounded away from zero. In either of the two cases, $\phi_{j,-}$ is a ψ-function. This is clear in the first case. In the second case, up to further partitioning the cell (as we have done for example in the proof of Proposition 5.9), we may suppose that $y > a(x)(\phi_{j,0}(x))^d$. We then modify ψ accordingly, by adding the bounded function $(\frac{1}{\phi_{j,0}})^d$ to $c_1(x), \dots, c_N(x)$ and considering the function $(\frac{a(x)(\phi_{j,0}(x))^d}{y})^{1/d}$ as the last component of ψ.

Therefore, $\exp(\mathrm{i}\phi_{j,-})$ is a complex-valued subanalytic function (see Definition 2.3) which can be expanded as a power series F in the variable $y^{-1/d}$ with analytic functions of x as coefficients. Let $K_j \in \mathbb{N}$ be such that, in the notation of (47), $r_j - \frac{K_j}{d} < -1$. We split the power series F into a polynomial part, by summing up to K_j, and a rest G. Therefore, we can replace T_j by a finite sum of terms of the form appearing in (47), but with the further property that $\phi_j \in \mathcal{S}(\Pi_m(A))[y^{1/d}]$, plus a final superintegrable term (corresponding to the rest G of the series).

Summing up, we have partitioned the index set J as $J^{\mathrm{int}} \cup J^{\mathrm{naive}}$, where T_j is superintegrable for every $j \in J^{\mathrm{int}}$ and, for all $j \in J^{\mathrm{naive}}$, T_j is of the form in (47) with $\phi_j \in \mathcal{S}(\Pi_m(A))[y^{1/d}]$. Now, by writing

$$\mathrm{e}^{\mathrm{i}\phi_j(x,y)} = \mathrm{e}^{\mathrm{i}(\phi_j(x,y) - \phi_j(x,0))} \mathrm{e}^{\mathrm{i}\phi_j(x,0)}$$

and absorbing $\mathrm{e}^{\mathrm{i}\phi_j(x,0)}$ into $f_j(x)$, we may assume that $\phi_j(x,0) = 0$ for all $j \in J^{\mathrm{naive}}$.

By further partitioning in x, we may also assume that, for all $j,k \in J^{\mathrm{naive}}$, $y \mapsto \phi_j(x,y)$ and $y \mapsto \phi_k(x,y)$ either define the same polynomial function for all $x \in \Pi_m(A)$ or define different polynomial functions for all $x \in \Pi_m(A)$. Therefore, by summing over terms for $j \in J^{\mathrm{naive}}$ with equal tuples $(r_j, s_j, \phi_j(x,y))$, we may assume that these tuples are distinct. We have thus completed the proof of Theorem 5.2. □

6. Proof of Theorem 2.20

In this section we complete the proof of Theorem 2.20 using Proposition 6.5(3) below, which states that, when we denote by f the function $\sum_{j\in J} c_j \mathrm{e}^{\mathrm{i}p_j(t)}$, there exists a real number $\varepsilon > 0$ such that the set $V_\varepsilon = \{t \in [0,+\infty) : |f(t)| \geq \varepsilon\}$ is not too sparse. To prove Proposition 6.5(3) let us first introduce a definition and notation. In what follows the notation vol_ℓ stands for the Lebesgue measure in the corresponding space $\mathbb{R}^\ell$, $\ell \geq 1$. All sets and maps involved with this notation are tacitly assumed to be Lebesgue measurable.

Definition 6.1

Let $\{x\} := x - \lfloor x \rfloor$ denote the fractional part of the real number x, and let $p = (p_1,\dots,p_\ell)\colon [0,+\infty) \to \mathbb{R}^\ell$ be a map. If $I_1,\dots,I_\ell \subseteq \mathbb{R}$ are bounded intervals with nonempty interior, we denote by I the box $\prod_{j=1}^{\ell} I_j$. For $T \geq 0$ we let

$$W_{p,I,T} := \big\{t \in [0,T] : \big\{p(t)\big\} \in I\big\},$$

where $\{p(t)\}$ denotes the tuple $(\{p_1(t)\},\dots,\{p_\ell(t)\})$.

The map p is said to be *continuously uniformly distributed modulo* 1 (*c.u.d. mod* 1) if, for every box $I \subseteq [0,1)^\ell$,

$$\lim_{T\to+\infty} \frac{\mathrm{vol}_1(W_{p,I,T})}{T} = \mathrm{vol}_\ell(I).$$

Remark 6.2

By [34] and [18, Corollary 9.1], a polynomial map $p = (p_1,\dots,p_\ell)$ is c.u.d. mod 1 provided that no nontrivial linear combination over $\mathbb{Z}$ of the polynomials p_j is constant.

LEMMA 6.3

Let $p\colon [0,+\infty) \to \mathbb{R}^\ell$ be a c.u.d. mod 1 map, let $I \subseteq [0,1)^\ell$ be a box, and let

$$W_{p,I} := \big\{t \in \mathbb{R} : \big\{p(t)\big\} \in I\big\}.$$

Then for all sufficiently large $k \in \mathbb{N}$,

$$\mathrm{vol}_1\big(W_{p,I} \cap [2^k, 2^{k+1}]\big) \geq 2^{k-1}\,\mathrm{vol}_\ell(I) \qquad \text{and} \qquad \int_{W_{p,I}} \frac{\mathrm{d}t}{t} = +\infty.$$

Proof

Let us denote $\mathrm{vol}_\ell(I)$ by s. By definition there exists $T_0 \geq 0$ such that, for every

$T \geq T_0$,

$$\mathrm{vol}_1(W_{p,I,T}) \in \Big[T\frac{5s}{6}, T\frac{7s}{6}\Big].$$

It follows that, for any $k \in \mathbb{N}$ such that $2^k \geq T_0$,

$$\mathrm{vol}_1(W_{p,I}) \cap [2^k, 2^{k+1}] = \mathrm{vol}_1(W_{p,I,2^{k+1}} \setminus W_{p,I,2^k}) \geq s\frac{5}{6}2^{k+1} - s\frac{7}{6}2^k = s2^{k-1}.$$

Therefore, denoting by k_0 the smallest integer k such that $2^k \geq T_0$, we have

$$\int_{W_{p,I}} \frac{\mathrm{d}t}{t} \geq \int_{W_{p,I} \cap [2^{k_0}, +\infty)} \frac{\mathrm{d}t}{t} = \sum_{k=k_0}^{+\infty} \int_{W_{p,I} \cap [2^k, 2^{k+1}]} \frac{\mathrm{d}t}{t} \geq s \sum_{k=k_0}^{+\infty} \frac{2^{k-1}}{2^{k+1}} = +\infty,$$

which concludes the proof. □

Remark 6.4
Let $c_1, \ldots, c_n \in \mathbb{C} \setminus \{0\}$, and let $p_1(t), \ldots, p_n(t) \in \mathbb{R}[t]$ be distinct polynomials such that $p_1(0) = \cdots = p_n(0) = 0$ and such that at least one of them is not constantly zero. Consider the function $f(t) = \sum_{j=1}^n c_j \mathrm{e}^{\mathrm{i}p_j(t)}$. Extract from the family of the polynomials p_j a basis of the $\mathbb{Q}$-vector space spanned by this family. Without loss of generality, we may suppose that such a basis is given by $(p_1, \ldots, p_\ell)$. Write

$$p_k = r_{k,1}p_1 + \cdots + r_{k,\ell}p_\ell, \quad k = \ell + 1, \ldots, n,$$

with $r_{k,j} \in \mathbb{Q}$ for $k = \ell + 1, \ldots, n$, $j = 1, \ldots, \ell$. If, for $j = 1, \ldots, \ell$, we denote by ρ_j the least common multiple of the denominators of the nonzero rational numbers among $r_{\ell+1,j}, \ldots, r_{n,j}$ and we let $\widetilde{p}_j = p_j/2\pi\rho_j$, then we have, for $k = \ell + 1, \ldots, n$,

$$p_k = 2\pi s_{k,1}\widetilde{p}_1 + \cdots + 2\pi s_{k,\ell}\widetilde{p}_\ell,$$

where $s_{k,1}, \ldots, s_{k,\ell} \in \mathbb{Z}$ and the family of polynomials $(\widetilde{p}_1, \ldots, \widetilde{p}_\ell)$ is independent over $\mathbb{Z}$. To sum up, one can write

$$f(t) = P(\mathrm{e}^{2\pi\mathrm{i}\widetilde{p}_1(t)}, \ldots, \mathrm{e}^{2\pi\mathrm{i}\widetilde{p}_\ell(t)}),$$

where P is a Laurent polynomial in $\mathbb{C}[X_1, \ldots, X_\ell, \frac{1}{X_1}, \ldots, \frac{1}{X_\ell}]$ which contains at least $\ell \geq 1$ monomials of the form $c_j X_j^{\rho_j}$, with $c_j \neq 0$ and $\rho_j \in \mathbb{N}$. Therefore, P is not a constant. (Note that we have not assumed that the function f is not constant.)

Now since the family $(\widetilde{p}_1, \ldots, \widetilde{p}_\ell)$ is independent over $\mathbb{Z}$ and since $p_1(0) = \cdots = p_\ell(0) = 0$, no nontrivial $\mathbb{Z}$-linear combination of $\widetilde{p}_1, \ldots, \widetilde{p}_\ell$ is constant. Thus, by Remark 6.2, the map $p = (\widetilde{p}_1, \ldots, \widetilde{p}_\ell)$ is c.u.d. mod 1.

PROPOSITION 6.5
Let $f:\mathbb{R}\to\mathbb{C}$ *be given by a finite sum*

$$f(t)=\sum_{j\in J}c_j\mathrm{e}^{\mathrm{i}p_j(t)},$$

where the $c_j\in\mathbb{C}\setminus\{0\}$ *and the* $p_j(t)$ *are distinct polynomials in* $\mathbb{R}[t]$*, vanishing at* 0*. Then one can find* $\varepsilon>0$ *such that*

(1) *There exist two sequences* $(t_{0,n})_{n\in\mathbb{N}}$ *and* $(t_{1,n})_{n\in\mathbb{N}}$*, which both tend to* $+\infty$*, such that* $\forall n\in\mathbb{N}$, $|f(t_{0,n})-f(t_{1,n})|\geq\varepsilon$*. In particular,* $\lim_{t\to+\infty}f(t)$ *exists if and only if* $p_j=0$ *for all* $j\in J$ *(i.e., if and only if* f *is a constant function).*

(2) *There exists a sequence* $(t_n)_{n\in\mathbb{N}}$ *which tends to* $+\infty$ *such that, for all* $n\geq 0$, $|f(t_n)|\geq\varepsilon$.

(3) $\int_{V_\varepsilon}\frac{1}{t}\,\mathrm{d}t=+\infty$*, where* $V_\varepsilon=\{t\in[1,+\infty):|f(t)|\geq\varepsilon\}$.

Proof
We may assume without loss of generality that J is $\{1,\dots,n\}$. By Remark 6.4, one can write $f(t)=P(\mathrm{e}^{2\pi\mathrm{i}\widetilde{p}_1(t)},\dots,\mathrm{e}^{2\pi\mathrm{i}\widetilde{p}_\ell(t)})$, where P is a nonconstant Laurent polynomial in $\mathbb{R}[X_1,\dots,X_\ell,\frac{1}{X_1},\dots,\frac{1}{X_\ell}]$ and $(\widetilde{p}_1,\dots,\widetilde{p}_\ell)$ is a c.u.d. mod 1 map. Set $a_1=\mathrm{e}^{2\pi\mathrm{i}\alpha_1},b_1=\mathrm{e}^{2\pi\mathrm{i}\beta_1},\dots,a_\ell=\mathrm{e}^{2\pi\mathrm{i}\alpha_\ell},b_\ell=\mathrm{e}^{2\pi\mathrm{i}\beta_\ell}$, where $\alpha_1,\beta_1,\dots,\alpha_\ell,\beta_\ell\in[0,1)$ are complex numbers such that

$$\bigl|P(a_1,\dots,a_\ell)-P(b_1,\dots,b_\ell)\bigr|\geq 3\varepsilon,$$

for some $\varepsilon>0$, and let us then consider the two sets

$$A=\Bigl\{t\in\mathbb{R}:\bigl(\{\widetilde{p}_1(t)\},\dots,\{\widetilde{p}_\ell(t)\}\bigr)\in\prod_{j=1}^{\ell}A_j\Bigr\},$$

$$B=\Bigl\{t\in\mathbb{R}:\bigl(\{\widetilde{p}_1(t)\},\dots,\{\widetilde{p}_\ell(t)\}\bigr)\in\prod_{j=1}^{n}B_\ell\Bigr\},$$

where $A_j\subseteq[0,1)$ is an interval centered at α_j and $B_j\subseteq[0,1)$ is an interval centered at β_j. If we denote by $h(t)$ the map $(\mathrm{e}^{\mathrm{i}\widetilde{p}_1(t)},\dots,\mathrm{e}^{\mathrm{i}\widetilde{p}_\ell(t)})$, since $(\widetilde{p}_1,\dots,\widetilde{p}_\ell)$ is a c.u.d. mod 1 map, by the continuity of h and P, by taking our intervals A_j and B_j sufficiently small, one can find two sequences $(t_{0,n})_{n\in\mathbb{N}}\in A$ and $(t_{1,n})_{n\in\mathbb{N}}\in B$ both tending to $+\infty$ such that

$$\forall n\in\mathbb{N},\quad \bigl|P\bigl(h(t_{0,n})\bigr)-P(a_1,\dots,a_\ell)\bigr|\leq\varepsilon \qquad\text{and}$$
$$\bigl|P\bigl(h(t_{1,n})\bigr)-P(b_1,\dots,b_\ell)\bigr|\leq\varepsilon.$$

This gives that, $\forall n \in \mathbb{N}$, $|f(t_{0,n}) - f(t_{1,n})| \geq \varepsilon$ and proves (1). To prove (2) we repeat the same argument as in (1): we choose complex numbers $a_1 = \mathrm{e}^{2\pi \mathrm{i}\alpha_1}, \ldots, a_\ell = \mathrm{e}^{2\pi \mathrm{i}\alpha_\ell}$, with $\alpha_1 \ldots, \alpha_\ell, \in [0,1)$, such that $|P(a_1, \ldots, a_\ell)| \geq 2\varepsilon$, and we define as above the corresponding sets $A_1, \ldots, A_\ell$ and A with the property that, when $t \in A$, $|f(t) - P(a_1, \ldots, a_\ell)| \leq \varepsilon$. One thus has that, for every $t \in A$, $|f(t)| \geq \varepsilon$. However, A certainly contains a sequence $(t_n)_{n\in\mathbb{N}}$ which tends to $+\infty$, since $(\widetilde{p_1}, \ldots, \widetilde{p_\ell})$ is a c.u.d. mod 1 map. This proves (2).

Now, since the set A defined above is such that $A \subseteq V_\varepsilon$ and since $(\widetilde{p_1}, \ldots, \widetilde{p_\ell})$ is c.u.d. mod 1, by Lemma 6.3 we have proved (3). □

We now complete the proof of Theorem 2.20.

Proof of Theorem 2.20

Let $f \in \mathcal{C}^{\exp}(X \times \mathbb{R})$, and apply Theorem 5.2 to f. This produces a finite partition $\mathcal{A}$ of $X \times \mathbb{R}$ into cells over $\mathbb{R}^m$. Consider one such cell $A \in \mathcal{A}$ that is open over $\mathbb{R}^m$, and let θ be a center for A. Write

$$f \circ P_\theta(x,y) = \sum_{j \in J^{\mathrm{int}}} T_j(x,y) + \sum_{j \in J^{\mathrm{naive}}} T_j(x,y) \quad \text{on } A_\theta.$$

Therefore,

$$f = \sum_{j \in J^{\mathrm{int}}} T_j \circ P_\theta^{-1} + \sum_{j \in J^{\mathrm{naive}}} T_j \circ P_\theta^{-1} \quad \text{on } A.$$

If $J^{\mathrm{naive}} = \emptyset$, then we are done. So suppose that $J^{\mathrm{naive}} \neq \emptyset$, which implies that A_θ is unbounded above (i.e., $b \equiv +\infty$, in the notation of Definition 3.4).

Recall from Remark 3.14(1) that

$$\partial_y P_\theta(y) := \frac{\partial P_{\theta,m+1}}{\partial y}(x,y) = \sigma\tau y^{\tau-1}$$

and that $\tau - 1$ equals either 0 or -2. Note that

$$\mathrm{Int}\big(T_j \circ P_\theta^{-1}, \Pi_m(A)\big) = \mathrm{Int}\big(T_j \partial_y P_\theta, \Pi_m(A)\big), \quad \forall j \in J.$$

For every $j \in J^{\mathrm{naive}}$, in the notation of (21), we have

$$T_j(x,y)\partial_y P_\theta(y) = \sigma\tau f_j(x) y^{r_j+\tau-1}(\log y)^{s_j} \mathrm{e}^{\mathrm{i}\phi_j(x,y)}, \tag{48}$$

which is integrable in y if and only if $f_j(x) = 0$ or $r_j + \tau < 0$. Therefore, by defining

$$J^{\mathrm{Int}} := J^{\mathrm{int}} \cup \{j \in J^{\mathrm{naive}} : r_j + \tau < 0\},$$

we see that, for each $j \in J$,

$$\operatorname{Int}\big(T_j \partial_y P_\theta, \Pi_m(A)\big) = \begin{cases} \Pi_m(A) & \text{if } j \in J^{\mathrm{Int}}, \\ \{x \in \Pi_m(A) : f_j(x) = 0\} & \text{if } j \in J^{\mathrm{naive}} := J \setminus J^{\mathrm{Int}}. \end{cases}$$

Let

$$g(x,y) = \sum_{j \in J^{\mathrm{Int}}} T_j \circ P_\theta^{-1}(x,y), \quad \text{for all } (x,y) \in A,$$

$$H = \bigcap_{j \in J^{\mathrm{naive}}} \big\{x \in \Pi_m(A) : f_j(x) = 0\big\}.$$

Note that, by taking the sum of the squares of the real and imaginary parts of f_j (see Remark 2.14(2)), we can write $H = \{x \in \Pi_m(A) : h(x) = 0\}$, for some $h \in \mathcal{C}^{\exp}(X)$.

It is clear that

$$\operatorname{Int}\big(g, \Pi_m(A)\big) = \Pi_m(A).$$

It remains to show that

$$\operatorname{Int}\big(f \restriction A, \Pi_m(A)\big) = H$$

and

$$f(x,y) = g(x,y) \quad \text{for all } (x,y) \in A \text{ with } x \in \operatorname{Int}\big(f \restriction A, \Pi_m(A)\big).$$

Clearly,

$$f(x,y) = g(x,y) \quad \text{for all } (x,y) \in A \text{ with } x \in H,$$

so $H \subseteq \operatorname{Int}(f \restriction A, \Pi_m(A))$.

To prove the other inclusion, we show that if $x \notin H$, then $x \notin \operatorname{Int}(\sum_{j \in J^{\mathrm{naive}}} T_j \partial_y P_\theta, \Pi_m(A)) = \operatorname{Int}(\sum_{j \in J^{\mathrm{naive}}} T_j \circ P_\theta^{-1}, \Pi_m(A))$, and hence $x \notin \operatorname{Int}(f \restriction A, \Pi_m(A))$. Fix $x \in \Pi_m(A) \setminus H$. Then the set $J := \{j \in J^{\mathrm{naive}} : f_j(x) \neq 0\} \subseteq J^{\mathrm{naive}}$ is nonempty (see (48)). Recall that the tuples $\{(r_j - \tau - 1, s_j, \phi_j(x,y))\}_{j \in J}$ are distinct and that $\rho_j := r_j - \tau - 1 \geq -1$ for all $j \in J$. Let

$$E = \big\{(\rho, s) \in \mathbb{Q} \times \mathbb{N} : \exists j \in J (\rho_j, s_j) = (\rho, s)\big\},$$

$$\text{and if } (\rho,s) \in E, \text{ let } E_{(\rho,s)} = \big\{j \in J : (\rho_j, s_j) = (\rho, s)\big\}.$$

Write

$$\begin{aligned} F(x,y) &= \sum_{j \in J} T_j \partial_y P_\theta(x,y) \\ &= \sum_{(\rho,s) \in E} y^\rho (\log y)^s \Big(\sum_{j \in E_{(\rho,s)}} \sigma \tau f_j(x) \mathrm{e}^{\mathrm{i}\phi_j(x,y)} \Big). \end{aligned}$$

Up to summing like terms, we may suppose that all polynomials ϕ_j in the previous sum are distinct. Let (ρ_0, s_0) be the lexicographic maximum of E, and let

$$G(x, y) = \sum_{j \in E_{(\rho_0, s_0)}} \sigma \tau f_j(x) \mathrm{e}^{\mathrm{i}\phi_j(x,y)}.$$

By applying Proposition 6.5(3) to the function $y \mapsto G(x, y)$, we obtain $\varepsilon > 0$ such that, for all $y \in V_\varepsilon$, $F(x, y)$ can be written as

$$\begin{aligned} y^{\rho_0} (\log y)^{s_0} G(x, y) \Big[1 + \sum_{(\rho,s) \in E \setminus \{(\rho_0, s_0)\}} y^{\rho - \rho_0} (\log y)^{s - s_0} \\ \times G(x, y)^{-1} \Big(\sum_{j \in E_{(\rho,s)}} \sigma \tau f_j(x) \mathrm{e}^{\mathrm{i}\phi_j(x,y)} \Big) \Big]. \end{aligned}$$

Note that there exists $M > 0$ such that for all $y \in V_\varepsilon \cap [M, +\infty)$ the square bracket in the previous equation is bounded from below by some positive constant K. Therefore,

$$\begin{aligned} \int_M^{+\infty} \big| F(x, y) \big| \, \mathrm{d}y &\geq \int_{V_\varepsilon \cap [M, +\infty)} \big| F(x, y) \big| \, \mathrm{d}y \\ &\geq \varepsilon K \int_{V_\varepsilon \cap [M, +\infty)} y^{\rho_0} (\log y)^{s_0} \, \mathrm{d}y \geq \varepsilon K \int_{V_\varepsilon \cap [M, +\infty)} \frac{1}{y} \, \mathrm{d}y. \end{aligned}$$

However, by Proposition 6.5(3) the last integral on the right diverges. Hence, $x \notin \mathrm{Int}(F, \Pi_m(A))$, and we are done. $\square$

7. Asymptotic expansions and limits

In this section we prove a series of consequences of our main results and their proofs. In Section 7.1 we prove that functions in $\mathcal{C}^{\exp}_{\text{naive}}$ have convergent asymptotic expansions of a certain form. We use this result to produce in Section 7.2 two examples of functions that are in $\mathcal{C}^{\exp}$ but not in $\mathcal{C}^{\exp}_{\text{naive}}$. In Section 7.3 we prove that $\mathcal{C}^{\exp}$ is stable under taking pointwise limits.

7.1. Asymptotic expansions of naive functions

Definition 7.1

A collection $\mathcal{G} = (g_n)_{n \in \mathbb{N}}$ of functions $g_n : (0, +\infty) \to \mathbb{R}$ with strictly positive germ at $+\infty$ is an *asymptotic scale at* $+\infty$ if, for all $n \in \mathbb{N}$, $\lim_{y \to +\infty} \frac{g_{n+1}(y)}{g_n(y)} = 0$. A $\mathbb{C}$-vector space $\mathcal{A}$ of functions $a : \mathbb{R} \to \mathbb{C}$ is a *space of coefficients* if for every $a \in \mathcal{A} \setminus \{0\}$ there are $\varepsilon > 0$ and a sequence $(y_n)_{n \in \mathbb{N}}$, with $\lim_{n \to +\infty} y_n = +\infty$, such that, $\forall n \in \mathbb{N}$, $|a(y_n)| > \varepsilon$.

Given a function $f : (0, +\infty) \to \mathbb{C}$, an asymptotic scale $\mathcal{G}$, and a space of coefficients $\mathcal{A}$, we say that f has a $(\mathcal{G}, \mathcal{A})$-*asymptotic expansion* at $+\infty$ if there are $y_0 > 0$

and a sequence $(a_n(y))_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that

$$\forall N \in \mathbb{N}\ \exists C > 0 \text{ such that, } \forall y > y_0, \quad \Big|f(y) - \sum_{n=0}^{N} a_n(y) g_n(y)\Big| \le C g_{N+1}(y).$$

LEMMA 7.2
If a function f admits a $(\mathcal{G},\mathcal{A})$-asymptotic expansion, then such an expansion is unique; that is, the sequence $(a_n(y))_{n\in\mathbb{N}}$ is uniquely determined.

Proof
Suppose that $(\widetilde{a}_n(y))_{n\in\mathbb{N}}$ is another sequence of coefficients. Supposing inductively that $a_n = \widetilde{a}_n$ for all $n < N$, we have

$$\begin{aligned}\big|\big(a_N(y) - \widetilde{a}_N(y)\big) g_N(y)\big| &= \Big|\sum_{n=0}^{N} a_n(y) g_n(y) - \sum_{n=0}^{N} \widetilde{a}_n(y) g_n(y)\Big| \\ &\le \Big|f(y) - \sum_{n=0}^{N} a_n(y) g_n(y)\Big| + \Big|f(y) - \sum_{n=0}^{N} \widetilde{a}_n(y) g_n(y)\Big| \\ &\le C_0 g_{N+1}(y),\end{aligned}$$

for some constant $C_0 > 0$ and for y sufficiently large. Dividing by $g_N(y)$, we obtain that $\lim_{y\to+\infty} |a_N(y) - \widetilde{a}_N(y)| = 0$. Now, the function $a(y) := a_N(y) - \widetilde{a}_N(y)$ belongs to $\mathcal{A}$, and if $a(y)$ is not identically zero, then $a(y)$ is bounded away from zero on some sequence of points going to $+\infty$. Hence, the only way for $a(y)$ to tend to zero is if $a(y)$ is identically zero. $\square$

PROPOSITION 7.3
Let $f \in \mathcal{C}^{\exp}_{\text{naive}}(\mathbb{R})$. Then f has a $(\mathcal{G},\mathcal{A})$-asymptotic expansion, where

- *$\mathcal{G} = (y^{r_n}(\log y)^{s_n})_{n\in\mathbb{N}}$ with $r_n \in \mathbb{Q}$, $s_n \in \mathbb{N}$, and $(r_n, s_n)_{n\in\mathbb{N}}$ a decreasing sequence of lexicographically ordered pairs;*
- *$\mathcal{A} = \{E(y) = \sum_{j\in J} c_j \mathrm{e}^{\mathrm{i} p_j(y^{1/d})} : (c_j)_{j\in J} \in \mathbb{C}^J\}$, for some $d \in \mathbb{N}$, some finite set $J \subseteq \mathbb{N}$, and for distinct polynomials $p_j(y) \in \mathbb{R}[y]$ with $p_j(0) = 0$.*

Moreover, if $E_n(y) = \sum_{j\in J} c_{j,n} \mathrm{e}^{\mathrm{i} p_j(y^{1/d})}$ are the coefficients of such an expansion, then for all sufficiently large y and for all $j \in J$, the series

$$f_j(y) = \sum_{n\in\mathbb{N}} c_{j,n} y^{r_n} (\log y)^{s_n} \tag{49}$$

will converge absolutely and

$$f(y) = \sum_{n\in\mathbb{N}} E_n(y) y^{r_n} (\log y)^{s_n}.$$

Proof

Note first that if $\mathcal{G}$ and $\mathcal{A}$ are as in the statement, then $\mathcal{G}$ is an asymptotic scale, and by Proposition 6.5(2), $\mathcal{A}$ is indeed a space of coefficients. By Remark 3.1, if $g \in \mathcal{S}(\mathbb{R})$, then, in the notation of (11), we have

$$\mathrm{e}^{\mathrm{i}g(y)} = G(y)\mathrm{e}^{\mathrm{i}p(y^{\frac{1}{d}})},$$

where $G(y) = \mathrm{e}^{\mathrm{i}g_0(y)}$ is a complex-valued subanalytic function (see Definition 2.3), since g_0 is bounded. Moreover, in the notation of (10),

$$\log g(y) = \log c + r \log y + h(y),$$

where $h(y) = \log(1 + H(y^{-\frac{1}{d}}))$ is in $\mathcal{S}([y_0, +\infty))$ for some sufficiently large y_0.

Hence, it is easy to see that if $f \in \mathcal{C}^{\exp}_{\mathrm{naive}}(\mathbb{R})$, then we may assume that, for y sufficiently large,

$$f(y) = \sum_{j \in J} f_j(y)\mathrm{e}^{\mathrm{i}p_j(y^{\frac{1}{d}})}, \tag{50}$$

where J is a finite set, f_j is a complex-valued constructible function, $d \in \mathbb{N}$, and $\{p_j(y) : j \in J\} \subseteq \mathbb{R}[y]$ is a collection of distinct polynomials such that $p_j(0) = 0$. Moreover, there exists a finite set K such that each f_j is of the form

$$f_j(y) = \sum_{k \in K} h_{j,k}(y)(\log y)^{s_{j,k}},$$

where $s_{j,k} \in \mathbb{N}$ and $h_{j,k}$ is a complex-valued subanalytic function.

Let us prove that f_j is indeed an absolutely convergent series. It is easy to see that Remark 3.1 also holds for complex-valued subanalytic functions (where now $c \in \mathbb{C}$ and H is a convergent power series with complex coefficients). Applying again Remark 3.1 to each $h_{j,k}$, for y sufficiently large we can write

$$f_j(y) = \sum_{k \in K} b_{j,k} y^{r_{j,k}} (\log y)^{s_{j,k}} \big(1 + H_{j,k}(y^{-\frac{1}{d}})\big), \tag{51}$$

where $b_{j,k} \in \mathbb{C}$, $r_{j,k} \in \mathbb{Q}$ is an integer multiple of $\frac{1}{d}$, and $H_{j,k}(y) = \sum_{m \in \mathbb{N}} a_{j,k,m} y^m$ is an absolutely convergent power series with complex coefficients and such that $H_{j,k}(0) = 0$. Hence, up to reorganizing the sum in (51), we have proved (49).

Now, setting $r_{j,k,m} = r_{j,k} - \frac{m}{d}$, we can write

$$f(y) = \sum_{\substack{(j,k) \in J \times K \\ m \in \mathbb{N}}} b_{j,k} a_{j,k,m} y^{r_{j,k,m}} (\log y)^{s_{j,k}} \mathrm{e}^{\mathrm{i}p_j(y^{\frac{1}{d}})}.$$

Let

$$I = \{(r,s) \in \mathbb{Q} \times \mathbb{N} : \exists j \in J, \exists k \in K, \exists m \in \mathbb{N} \text{ such that } (r_{j,k,m}, s_{j,k}) = (r,s)\},$$

$$\text{and if } (r,s) \in I, \text{ then let } I_{(r,s)} = \{(j,k,m) \in J \times K \times \mathbb{N} : (r_{j,k,m}, s_{j,k}) = (r,s)\}.$$

We can write

$$f(y) = \sum_{(r,s)\in I} y^r (\log y)^s E_{(r,s)}(y),$$

where $E_{(r,s)}(y) = \sum_{(j,k,m)\in I_{(r,s)}} b_{j,k} a_{j,k,m} \mathrm{e}^{\mathrm{i} p_j (y^{\frac{1}{d}})}$. Note that for every $(r,s) \in I$ the set $I_{(r,s)}$ is finite, so $E_{(r,s)}$ is a finite sum of exponentials. Moreover, if I_0 is the set of all $r \in \mathbb{Q}$ such that there exists $s \in \mathbb{N}$ with $(r,s) \in I$, then we have that I_0 is bounded from above (by $\max_{(j,k)\in J\times K} r_{j,k}$) and for every $r \in I_0$ there are finitely many $s \in \mathbb{N}$ such that $(r,s) \in I$. (In fact, the cardinality of the set of all s's such that there exists $r \in I_0$ with $(r,s) \in I$ is uniformly bounded by the product of the cardinalities of J and K.) Hence, with respect to the lexicographic order, I has the same order type as ω, and we can fix a decreasing bijection $\mathbb{N} \ni n \mapsto (r_n, s_n) = (r,s) \in I$.

Let us thus rename $E_n(y) = E_{(r_n,s_n)}(y)$. We have proved that, for y sufficiently large,

$$f(y) = \sum_{n\in\mathbb{N}} E_n(y) y^{r_n} (\log y)^{s_n}.$$

In particular, f indeed has a $(\mathcal{G}, \mathcal{A})$-asymptotic expansion. □

7.2. Two functions which are in $\mathcal{C}^{\exp}(\mathbb{R})$ but not in $\mathcal{C}^{\exp}_{\text{naive}}(\mathbb{R})$

Example 7.4
Consider the function $f(y) = \mathrm{e}^{-|y|}$.

Consider the Fourier transform of f:

$$\hat{f}(y) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} x y} \mathrm{e}^{-|x|} \, \mathrm{d}x.$$

It is well known that $\hat{f}$ is a semialgebraic integrable function, namely, $\hat{f}(y) = \frac{2}{1+4\pi^2 y^2}$ (see, e.g., [13]). Since we can compute f as the inverse Fourier transform of $\hat{f}$ and since $\hat{f}$ is semialgebraic, we have that f belongs to the class $\mathcal{C}^{\exp}(\mathbb{R})$. It follows from Remark 2.8 that if $g \in \mathcal{S}(\mathbb{R})$, then $\mathrm{e}^{-|g(y)|} \in \mathcal{C}^{\exp}(\mathbb{R})$. (In particular, $\mathrm{e}^{-y^2} \in \mathcal{C}^{\exp}(\mathbb{R})$.)

CLAIM

The function $f(y) = \mathrm{e}^{-|y|}$ *is not in* $\mathcal{C}^{\exp}_{\text{naive}}(\mathbb{R})$.

Proof

Suppose for a contradiction that $f \in \mathcal{C}^{\exp}_{\text{naive}}(\mathbb{R})$. By Proposition 7.3 we may write $f(y)$ as the sum of a convergent series

$$f(y) = \sum_{n \in \mathbb{N}} E_n(y) y^{r_n} (\log y)^{s_n}$$

for all sufficiently large y. Since the germ of f at $+\infty$ is nonzero, this series contains a nonzero term. Choose the least $n_0 \in \mathbb{N}$ such that $E_{n_0}(y)$ is not identically 0. Thus, there exists a constant $C > 0$ such that

$$\left| f(y) - E_{n_0}(y) y^{r_{n_0}} (\log y)^{s_{n_0}} \right| \le C y^{r_{n_0+1}} (\log y)^{s_{n_0+1}}$$

for all sufficiently large y. Since $\frac{f(y)}{y^{r_{n_0}} (\log y)^{s_{n_0}}}$ and $\frac{y^{r_{n_0+1}} (\log y)^{s_{n_0+1}}}{y^{r_{n_0}} (\log y)^{s_{n_0}}}$ both tend to 0 as $y \to +\infty$, dividing both sides of this inequality by $y^{r_{n_0}} (\log y)^{s_{n_0}}$ and letting y tend to $+\infty$ gives $\lim_{y \to +\infty} E_{n_0}(y) = 0$, which contradicts Proposition 6.5(2). $\square$

Example 7.5

Consider the sine integral $\mathrm{Si}\colon [0, +\infty) \to \mathbb{R}$, which is defined by

$$\mathrm{Si}(y) = \int_0^y \frac{\sin(t)}{t} \, \mathrm{d}t = \int_0^y \frac{\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}}{2\mathrm{i}t} \, \mathrm{d}t.$$

Clearly, $\mathrm{Si} \in \mathcal{C}^{\exp}([0, +\infty))$.

CLAIM

The function $\mathrm{Si}(y)$ *is not in* $\mathcal{C}^{\exp}_{\text{naive}}([0, +\infty))$.

Proof

Recall the classical asymptotic formula (see [1]):

$$\mathrm{Si}(y) \sim \frac{\pi}{2} - \frac{\cos y}{y} \sum_{k \in \mathbb{N}} (-1)^k \frac{(2k)!}{y^{2k}} - \frac{\sin y}{y} \sum_{k \in \mathbb{N}} (-1)^k \frac{(2k+1)!}{y^{2k+1}}.$$

Hence, $\mathrm{Si}(y)$ has a $(\mathcal{G}, \mathcal{A})$-asymptotic expansion, with $\mathcal{G}$ and $\mathcal{A}$ as in the statement of Proposition 7.3. However, in the notation of (49), the series $F_1(y) = \sum_{k \in \mathbb{N}} (-1)^k \frac{(2k)!}{y^{2k+1}}$ and $F_2(y) = \sum_{k \in \mathbb{N}} (-1)^k \frac{(2k+1)!}{y^{2k+2}}$ are divergent. Therefore, by Lemma 7.2 and Proposition 7.3, $\mathrm{Si}(y) \notin \mathcal{C}^{\exp}_{\text{naive}}([0, +\infty))$. $\square$

7.3. Pointwise limits

Definition 7.6
For any $X \subseteq \mathbb{R}^m$ and $f : X \times \mathbb{R} \to \mathbb{C}$, let

$$\operatorname{Lim}(f, X) := \left\{x \in X : \lim_{y\to+\infty} f(x, y) \text{ exists}\right\}.$$

PROPOSITION 7.7
Let $f \in \mathcal{C}^{\exp}(X \times \mathbb{R})$ for some subanalytic set $X \subseteq \mathbb{R}^m$. There exist $g, h \in \mathcal{C}^{\exp}(X)$ such that

$$\operatorname{Lim}(f, X) = \left\{x \in X : h(x) = 0\right\}$$

and such that, for all $x \in \operatorname{Lim}(f, X)$,

$$\lim_{y\to+\infty} f(x, y) = g(x).$$

Proof
Apply Theorem 5.2 to $f(x, y)$ with respect to y. Focus on one cell of the form

$$A = \left\{(x, y) : x \in \Pi_m(A), y > a(x)\right\}.$$

Let

$$\begin{aligned} &E = \left\{(r, s) \in \mathbb{Q} \times \mathbb{N} : \exists j \in J (r_j, s_j) = (r, s)\right\}, \\ &\text{and if } (r, s) \in E, \text{ let } E_{(r,s)} = \left\{j \in J : (r_j, s_j) = (r, s)\right\}. \end{aligned} \tag{52}$$

The terms in the preparation involving $y^r(\log y)^s$ with $r < 0$ may be neglected since they affect neither the existence of $\lim_{y\to+\infty} f(x, y)$ nor its value when it exists. So we may assume that $f(x, y)$ is naive in y with nonnegative powers of y in each term of the preparation. Write f as the finite sum

$$f(x, y) = \sum_{(r,s)\in E} y^r(\log y)^s\Big(\sum_{j\in E_{(r,s)}} f_j(x)\mathrm{e}^{\mathrm{i}\phi_j(x,y)}\Big), \tag{53}$$

where each f_j is in $\mathcal{C}^{\exp}(\Pi_m(A))$ and where we have that, for each (r, s) and each $x \in \Pi_m(A)$,

$$\phi_j(x, y) = \sum_{k=1}^{m} a_{j,k}(x) y^{\frac{k}{d}} \quad \text{for } j \in E_{(r,s)}$$

is a family of distinct polynomials in $y^{\frac{1}{d}}$ with subanalytic coefficients $a_{j,k}$. By partitioning in x we may also assume that if there exist $\widetilde{j} \in E_{(r,s)}$ and $\widetilde{x} \in \Pi_m(A)$ such that $\phi_{\widetilde{j}}(\widetilde{x}, y) = 0$ for all y such that $(\widetilde{x}, y) \in A$, then $\phi_{\widetilde{j}}(x, y) = 0$ for all $(x, y) \in A$. (Note that there is at most one such $\widetilde{j} \in E_{(r,s)}$ such that $\phi_{\widetilde{j}} \equiv 0$ because, for each $x \in \Pi_m(A)$, $(\phi_j(x, y))_{j \in J}$ is a family of distinct polynomials in $y^{\frac{1}{d}}$.)

CLAIM

For each $x \in \Pi_m(A)$*,* $x \in \mathrm{Lim}(f, \Pi_m(A))$ *if and only if the following two conditions hold:*

(1) *for each* $(r, s) \in E$ *such that* $r > 0$ *or* $s > 0$*, we have that* $f_j(x) = 0$ *for all* $j \in E_{(r,s)}$*;*

(2) *for all* $j \in E_{(0,0)}$ *such that* $\phi_j \not\equiv 0$*, we have* $f_j(x) = 0$.

Proof

To prove the claim, fix $x \in \Pi_m(A)$. Observe that if conditions (1) and (2) hold, then either f is identically 0, or else there exists $j_0 \in E_{(0,0)}$ such that $f(x, y) = f_{j_0}(x)$ for all y. Either way, $\lim_{y \to +\infty} f(x, y)$ exists trivially.

To prove the converse, assume that $x \in \mathrm{Lim}(f, \Pi_m(A))$. Conditions (1) and (2) clearly hold if $f_j(x) = 0$ for all $j \in \bigcup_{(r,s) \in E} E_{(r,s)}$, so assume otherwise. Choose (r_0, s_0) maximal with respect to the lexicographical ordering such that $f_j(x) \neq 0$ for some $j \in E_{(r_0,s_0)}$. By Proposition 6.5(2), since $\lim_{y \to +\infty} f(x, y)$ exists, it follows that $r_0 = s_0 = 0$. Thus, condition (1) holds, and we have

$$f(x, y) = \sum_{j \in E_{(0,0)}} f_j(x) \mathrm{e}^{\mathrm{i}\phi_j(x,y)}$$

for all y. Proposition 6.5(1) now shows that condition (2) holds. This proves the claim. $\square$

The claim easily implies the proposition. Indeed, define

$$h = \Big(\sum_{\substack{(r,s) \in E \text{ such that} \\ r > 0 \text{ or } s > 0}} \sum_{j \in E_{(r,s)}} |f_j|^2 \Big) + \Big(\sum_{\substack{j \in E_{(0,0)} \text{ such that} \\ \phi_j \not\equiv 0}} |f_j|^2 \Big);$$

define $g = f_{j_0}$ if there exists $j_0 \in E_{(0,0)}$ such that $\phi_{j_0} \equiv 0$, and define $g = 0$ otherwise. Then $g, h \in \mathcal{C}^{\exp}(\Pi_m(A))$. The claim shows that

$$\mathrm{Lim}\big(f \restriction A, \Pi_m(A)\big) = \big\{x \in A : h(x) = 0\big\}$$

and that

$$f(x, y) = g(x) \quad \text{for all } (x, y) \in A \text{ such that } h(x) = 0. \qquad \square$$

8. Parametric L^p-completeness and the Fourier–Plancherel transform

In this section we prove a parametric L^p-completeness theorem for $\mathcal{C}^{\exp}$ and use this to show that $\mathcal{C}^{\exp}$ is closed under the Fourier–Plancherel transform.

Definition 8.1
Let $X \subseteq \mathbb{R}^m$, let $f : X \times \mathbb{R} \to \mathbb{C}$ be Lebesgue measurable, and let $p \in [1, +\infty]$. For each $y \in \mathbb{R}$, define $f_y : X \to \mathbb{C}$ by $f_y(x) = f(x, y)$ for all $x \in X$. We say that the family of functions $(f_y)_{y\in\mathbb{R}}$ is *Cauchy* in $L^p(X)$ as $y \to +\infty$ if $(f_y)_{y\in\mathbb{R}} \subseteq L^p(X)$ and for all $\varepsilon > 0$ there exists $y_0 \in \mathbb{R}$ such that

$$\|f_y - f_{y'}\|_p < \varepsilon \quad \text{for all } y, y' \geq y_0.$$

PROPOSITION 8.2
Let $p \in [1, +\infty]$ and $f \in \mathcal{C}^{\exp}(X \times \mathbb{R})$ for a subanalytic set $X \subseteq \mathbb{R}^m$, and suppose that $(f_y)_{y\in\mathbb{R}}$ is Cauchy in $L^p(X)$ as $y \to +\infty$. Then there exist $g \in \mathcal{C}^{\exp}(X) \cap L^p(X)$ and a subanalytic set $X_0 \subseteq X$ such that $\mathrm{vol}_m(X \setminus X_0) = 0$*,*

$$\lim_{y\to+\infty} \|f_y - g\|_p = 0,$$

and

$$\lim_{y\to+\infty} f(x, y) = g(x) \quad \textit{for all } x \in X_0.$$

Before proving Proposition 8.2, we use it to show that $\mathcal{C}^{\exp}$ is closed under the Fourier–Plancherel transform.

THEOREM 8.3
Let $\widetilde{\mathscr{F}}$ be the Fourier–Plancherel extension of the Fourier transform to $L^2(\mathbb{R}^n)$, as in (6). Then, the image of $\mathcal{C}^{\exp}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ under $\widetilde{\mathscr{F}}$ is $\mathcal{C}^{\exp}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Proof
Let $f \in \mathcal{C}^{\exp}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. We use coordinates $x = (x_1, \dots, x_n)$ and $t = (t_1, \dots, t_n)$ on $\mathbb{R}^n$. For each $y \in \mathbb{R}$, define

$$B_y = \{t \in \mathbb{R}^n : |t| \leq y\},$$

and observe that $L^2(B_y) \subseteq L^1(B_y)$ for each y (by the Cauchy–Schwartz inequality, since $\mathrm{vol}_n(B_y) < +\infty$). So we may define $F : \mathbb{R}^{n+1} \to \mathbb{C}$ by

$$F(x, y) := \int_{B_y} f(t) \mathrm{e}^{-2\pi \mathrm{i} t\cdot x} \, \mathrm{d}t = \int_{\mathbb{R}^n} \chi_{B_y}(t) f(t) \mathrm{e}^{-2\pi \mathrm{i} t\cdot x} \, \mathrm{d}t, \tag{54}$$

and we have that $F \in \mathcal{C}^{\exp}(\mathbb{R}^{n+1})$, since $\mathcal{C}^{\exp}$ is closed under integration. The extended Fourier transform $\widetilde{\mathscr{F}}(f)$ is the equivalence class of functions $[\widehat{f}]$ (with respect to almost everywhere equivalence) that is defined by the condition

$$\lim_{y \to +\infty} \|\widehat{f} - F_y\|_2 = 0.$$

Thus, $(F_y)_{y \in \mathbb{R}}$ is Cauchy in $L^2(\mathbb{R}^n)$ as $y \to +\infty$, so by Proposition 8.2 we may fix $g \in \mathcal{C}^{\exp}(\mathbb{R}^n)$ such that

$$\lim_{y \to +\infty} \|g - F_y\|_2 = 0,$$

and hence $[\widehat{f}] = [g]$. This shows that the extended Fourier transform $\widetilde{\mathscr{F}}$ maps $\mathcal{C}^{\exp}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ into $\mathcal{C}^{\exp}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. A completely symmetric argument, where one simply replaces i with $-$i in (54), shows that the inverse extended Fourier transform maps $\mathcal{C}^{\exp}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ into $\mathcal{C}^{\exp}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ as well, so $\mathcal{C}^{\exp}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is in fact the image of $\mathcal{C}^{\exp}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ under $\widetilde{\mathscr{F}}$. $\square$

The remainder of this section is devoted to the proof of Proposition 8.2, which requires us to develop a bit of machinery. This proof is somewhat similar to the proof of Proposition 7.7, except we cannot rely on the facts about c.u.d. mod 1 maps quoted in Remark 6.2. Instead, we need to adapt these facts to parametric families of maps that are c.u.d. mod 1 in a certain uniform sense. We are not aware of a reference in the literature on c.u.d. mod 1 maps that considers this parametric case, so this section develops this material from scratch. We remark that the proofs of Lemma 8.6 and Proposition 8.7 below use ideas found in the proofs of [18, Example 9.2 and the closely interrelated Theorems 1.1, 2.1, 6.1, 6.2, 9.1, 9.2, and 9.9].

Let us first give the parametric version of Definition 6.1. For this, let X be a nonempty set, and let $\psi = (\psi_1, \dots, \psi_n) : X \times [0, +\infty) \to \mathbb{R}^n$ be a map. If $I_1, \dots, I_n \subseteq \mathbb{R}$ are bounded intervals with nonempty interior, we denote by I the box $\prod_{j=1}^{\ell} I_j$, and for $T \geq 0$ and $x \in X$, we let

$$W^x_{\psi,I,T} := \{t \in [0, T] : \{\psi(x,t)\} \in I\},$$

where $\{\psi(x,t)\}$ denotes the vector of fractional parts $(\{\psi_1(x,t)\}, \dots, \{\psi_n(x,t)\})$ of the components of ψ.

Definition 8.4
With this notation, we say that the map ψ is *continuously uniformly distributed modulo* 1 on X (abbreviated as *c.u.d. mod* 1 on X) if, for every box $I \subseteq [0,1)^n$,

$$\lim_{T \to +\infty} \sup_{x \in X} \frac{\mathrm{vol}_1(W^x_{\psi,I,T})}{T} = \mathrm{vol}_n(I).$$

The following remark is the parametric analogue of Lemma 6.3.

Remark 8.5
Suppose that $\psi : X \times [0,+\infty) \to \mathbb{R}^n$ is c.u.d. mod 1 on X. Then for each box $I \subseteq [0,1)^n$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$ and for all $x \in X$,

$$\mathrm{vol}_1\big(\big\{t \in [2^k, 2^{k+1}\big) : \big\{\psi(x,t)\big\} \in I\big\}) \geq 2^{k-1}\,\mathrm{vol}_n(I).$$

This bound is proven just as Lemma 6.3, using the uniform limit in the parameter x provided by Definition 8.4.

The following technical lemma will be used in the proof of the forthcoming Proposition 8.7.

LEMMA 8.6
Define $\phi : X \times [0,+\infty) \to \mathbb{R}$ by

$$\phi(x,t) = \sum_{j=0}^{d} \phi_j(x) t^j,$$

where d is a positive integer, the functions $\phi_0, \ldots, \phi_d : X \to \mathbb{R}$ are bounded, and there exists $\varepsilon > 0$ such that $|\phi_d(x)| > \varepsilon$ for all $x \in X$. Then the function $\Phi : X \times [0,+\infty) \to \mathbb{C}$ defined by

$$\Phi(x,T) = \int_0^T \mathrm{e}^{\mathrm{i}\phi(x,t)}\,\mathrm{d}t$$

is bounded.

Proof
It suffices to show that for some suitable choice of $T_0 \geq 0$ there exists a constant $C > 0$ such that, for all $(x,T) \in X \times [T_0,+\infty)$,

$$\left|\int_{T_0}^T \mathrm{e}^{\mathrm{i}\phi(x,t)}\,\mathrm{d}t\right| \leq C.$$

Define

$$f(x,t) := \frac{\phi(x,t)}{\phi_d(x)} = t^d + \sum_{j=0}^{d-1} \frac{\phi_j(x)}{\phi_d(x)} t^j,$$

and observe that our assumed bounds on $\phi_0, \ldots, \phi_n$ show that the coefficient functions in x of the polynomial $f(x,t)$ are bounded. Therefore, by computing $\frac{\partial f}{\partial t}$ and $\frac{\partial^2 f}{\partial t^2}$ and factoring out their leading terms, we may fix $T_0 > 0$ such that

$$\begin{aligned}
&\frac{\partial f}{\partial t}(x,t) = dt^{d-1}u(x,t) \qquad \text{and} \\
&\frac{\partial^2 f}{\partial t^2}(x,t) = \begin{cases} 0 & \text{if } d = 1, \\ d(d-1)t^{d-2}v(x,t) & \text{if } d > 1, \end{cases}
\end{aligned} \tag{55}$$

for some functions $u(x,t)$ and $v(x,t)$ (when $d > 1$) that take values in $[\frac{1}{2}, \frac{3}{2}]$ for all $(x,t) \in X \times [T_0, +\infty)$. Therefore, $\frac{\partial f}{\partial t} > 0$ and $\frac{\partial^2 f}{\partial t^2} \geq 0$ on $X \times [T_0, +\infty)$, so for each $x \in X$, the functions $t \mapsto f(x,t)$ and $t \mapsto \frac{\partial f}{\partial t}(x,t)$ are, respectively, strictly increasing and monotonically increasing on $[T_0, +\infty)$. For each $x \in X$, let $t = g(x,s)$ be the inverse of $s = f(x,t)$, where $t \geq T_0$ and $s \geq f(x, T_0)$. For each $T \geq T_0$, we can perform the integral substitution

$$s = f(x,t), \quad \mathrm{d}s = \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}t = \frac{\partial f}{\partial t}\big(x, g(x,s)\big)\,\mathrm{d}t$$

to write

$$\begin{aligned}
\int_{T_0}^{T} \mathrm{e}^{\mathrm{i}\phi(x,t)}\,\mathrm{d}t &= \int_{T_0}^{T} \mathrm{e}^{\mathrm{i}\phi_d(x) f(x,t)}\,\mathrm{d}t \\
&= \int_{f(x,T_0)}^{f(x,T)} \frac{\mathrm{e}^{\mathrm{i}\phi_d(x)s}}{\frac{\partial f}{\partial t}(x, g(x,s))}\,\mathrm{d}s.
\end{aligned} \tag{56}$$

The function

$$s \mapsto \frac{1}{\frac{\partial f}{\partial t}(x, g(x,s))}$$

is monotonically decreasing on $[f(x,T_0), +\infty)$, so we can apply the second mean value theorem for integrals to the real and complex parts of the integral (56). For the real part, this gives

$$\int_{f(x,T_0)}^{f(x,T)} \frac{\cos(\phi_d(x)s)}{\frac{\partial f}{\partial t}(x, g(x,s))}\,\mathrm{d}s = \frac{1}{\frac{\partial f}{\partial t}(x, T_0)} \int_{f(x,T_0)}^{\xi(x,T)} \cos\big(\phi_d(x)s\big)\,\mathrm{d}s \tag{57}$$

for some $\xi(x,T) \in (f(x,T_0), f(x,T))$. Since $s \mapsto \cos(\phi_d(x)s)$ has an antiderivative with period $\frac{2\pi}{|\phi_d(x)|}$ and since $\frac{2\pi}{|\phi_d(x)|} \leq \frac{2\pi}{\varepsilon}$, the integral on the right-hand side of (57) may be replaced with an integral over an interval of length at most $\frac{2\pi}{\varepsilon}$. This, along with the form of $\frac{\partial f}{\partial t}$ given in (55), shows that (57) is bounded. A nearly identical calculation shows the same for the imaginary part of (56), and the lemma follows. $\square$

The following Proposition 8.7 is the parametric analogue of Remark 6.2, stating that polynomials maps are c.u.d. mod 1 when nontrivial $\mathbb{Z}$-linear combinations of their components are nonconstant. For technical reasons, in the parametric case it is more convenient to reduce to the situation of maps with monomial instead of polynomial components.

PROPOSITION 8.7
Consider a map $\psi = (\psi_1, \dots, \psi_n) : X \times [0, +\infty) \to \mathbb{R}^n$, where X is a compact topological space and where, for each $j \in \{1, \dots, n\}$,

$$\psi_j(x,t) = g_j(x)t^{\gamma_j}$$

for some continuous function $g_j : X \to \mathbb{R}$ and positive integer γ_j. Assume that, for each $x \in X$, the functions $t \mapsto \psi_1(x,t), \dots, t \mapsto \psi_n(x,t)$ are linearly independent over $\mathbb{Q}$. Then ψ is c.u.d. mod 1 on X.

The following notation and observation will be used in the proof of Proposition 8.7.

Remark 8.8
Let $\gamma = \max\{\gamma_1, \dots, \gamma_n\}$, and for each $k \in \{1, \dots, \gamma\}$, let $J_k = \{j \in \{1, \dots, n\} : \gamma_j = k\}$. The assumption that $t \mapsto \psi_1(x,t), \dots, t \mapsto \psi_n(x,t)$ are linearly independent over $\mathbb{Q}$ for each $x \in X$ is equivalent to saying that, for each $k \in \{1, \dots, \gamma\}$ and $x \in X$, the family of real numbers $(g_j(x))_{j \in J_k}$ is linearly independent over $\mathbb{Q}$.

Proof of Proposition 8.7
We will use the variables t, $y = (y_1, \dots, y_n)$, and $z = (z_1, \dots, z_n)$, and write $\mathrm{d}y$ for $\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$. Let $\varepsilon > 0$ and a box $I = \prod_{j=1}^n I_j \subseteq [0,1)^n$ be given. For each $j \in \{1, \dots, n\}$, let $\chi_{I_j} : \mathbb{R} \to \{0, 1\}$ be the 1-periodic extension of the characteristic function of I_j in $[0, 1)$, and define $\chi_I : \mathbb{R}^n \to \{0, 1\}$ by $\chi_I(y) = \prod_{j=1}^n \chi_{I_j}(y_j)$. Thus,

$$\mathrm{vol}_1\big(\big\{t \in [0,T] : \big\{\psi(x,t)\big\} \in I\big\}\big) = \int_0^T \chi_I \circ \psi(t)\,\mathrm{d}t.$$

Let $\varepsilon > 0$, and fix $\delta \in (0,1]^n$ sufficiently small so that $1 - (1-\delta)^n < \frac{\varepsilon}{4}$. For each $j \in \{1, \dots, n\}$, fix 1-periodic continuous functions $p_j : \mathbb{R} \to [0,1]$ and $q_j : \mathbb{R} \to [0,1]$ such that $p_j(t) \le \chi_{I_j}(t) \le q_j(t)$ for all $t \in \mathbb{R}$ and such that

$$\mathrm{vol}_1\big(\big\{t \in [0,1] : p_j(t) \neq \chi_{I_j}(t)\big\}\big) \le \delta,$$
$$\mathrm{vol}_1\big(\big\{t \in [0,1] : q_j(t) \neq \chi_{I_j}(t)\big\}\big) \le \delta.$$

Define $p : \mathbb{R}^n \to [0,1]$ and $q : \mathbb{R}^n \to [0,1]$ by $p(y) = \prod_{j=1}^n p_j(y_j)$ and $q(y) = \prod_{j=1}^n q_j(y_j)$. Since $p(y) \le \chi_I(y) \le q(y)$ for all $y \in \mathbb{R}^n$, we have, for all $x \in X$,

$$\frac{1}{T}\int_0^T p \circ \psi(x,t)\,\mathrm{d}t \le \frac{1}{T}\int_0^T \chi_I \circ \psi(x,t)\,\mathrm{d}t \le \frac{1}{T}\int_0^T q \circ \psi(x,t)\,\mathrm{d}t. \tag{58}$$

It now suffices to show that there exists $T_0 > 0$ such that the lower and upper bounds in (58) are within ε of $\mathrm{vol}_n(I)$ for all $x \in X$ and $T \ge T_0$. The computations involving the lower bound and the upper bound are identical, so we only show the computation with the lower bound.

Fix $\eta \in (0,1]^n$ sufficiently small so that, for all $y, z \in [-2,2]^n$, if $|y_j - z_j| < \eta$ for all $j \in \{1,\dots,n\}$, then $|\prod_{j=1}^n y_j - \prod_{j=1}^n z_j| < \frac{\varepsilon}{4}$. By a Weierstrass approximation theorem, for each $j \in \{1,\dots,n\}$ we may fix a trigonometric polynomial

$$P_j(t) = \sum_{\alpha=-N_j}^{N_j} c_{j,\alpha}\mathrm{e}^{2\pi\mathrm{i}\alpha t}$$

(where $N_j \in \mathbb{N}$ and $c_{j,\alpha} \in \mathbb{C}$ for each α) such that

$$\bigl|p_j(t) - P_j(t)\bigr| \le \eta \tag{59}$$

for all $t \in \mathbb{R}$. Define $P : \mathbb{R}^n \to \mathbb{C}$ by $P(y) = \prod_{j=1}^n P_j(y_j)$. Since $\mathrm{vol}_n(I) = \int_{[0,1]^n} \chi_I(y)\,\mathrm{d}y$, we have

$$\begin{aligned}
&\Bigl|\frac{1}{T}\int_0^T p \circ \psi(x,t)\,\mathrm{d}t - \mathrm{vol}_n(I)\Bigr| \\
&\quad\le \Bigl|\frac{1}{T}\int_0^T \bigl(p \circ \psi(x,t) - P \circ \psi(x,t)\bigr)\,\mathrm{d}t\Bigr| && (*)\\
&\qquad + \Bigl|\frac{1}{T}\int_0^T P \circ \psi(x,t)\,\mathrm{d}t - \int_{[0,1]^n} P(y)\,\mathrm{d}y\Bigr| && (**)\\
&\qquad + \Bigl|\int_{[0,1]^n} \bigl(P(y) - p(y)\bigr)\,\mathrm{d}y\Bigr| && (***)\\
&\qquad + \Bigl|\int_{[0,1]^n} \bigl(p(y) - \chi_I(y)\bigr)\,\mathrm{d}y\Bigr|. && (****)
\end{aligned} \tag{60}$$

Note that $|p_j(t)| \le 1$ and $|P_j(t)| \le |p_j(t)| + |P_j(t) - p_j(t)| \le 1 + \eta \le 2$ for all $j \in \{1,\dots,n\}$ and $t \in \mathbb{R}$, so by our choice of η, (59) implies that $|p(y) - P(y)| \le \frac{\varepsilon}{4}$ for all $y \in \mathbb{R}^n$. Therefore, the terms (*) and (***) in (60) are both bounded above by $\frac{\varepsilon}{4}$. And since

$$\bigcap_{j=1}^n \bigl\{y \in [0,1]^n : p_j(y_j) = \chi_{I_j}(y_j)\bigr\} \subseteq \bigl\{y \in [0,1]^n : p(y) = \chi_I(y)\bigr\}$$

and

$$\mathrm{vol}_n\Big(\bigcap_{j=1}^{n}\big\{y \in [0,1]^n : p_j(y_j) = \chi_{I_j}(y_j)\big\}\Big) \geq (1-\delta)^n,$$

it follows that

$$\mathrm{vol}_n\big(\big\{y \in [0,1]^n : p(y) \neq \chi_I(y)\big\}\big) \leq 1-(1-\delta)^n \leq \frac{\varepsilon}{4},$$

so the term (****) in (60) is also bounded above by $\frac{\varepsilon}{4}$. So to finish, we need to show that there exists $T_0 > 0$ such that the term (**) in (60) is also bounded above by $\frac{\varepsilon}{4}$ for all $x \in X$ and all $T \geq T_0$.

We have

$$P(y) = \prod_{j=1}^{n}\Big(\sum_{\alpha_j=-N_j}^{N_j} c_{j,\alpha_j}\mathrm{e}^{2\pi\mathrm{i}\alpha_j y_j}\Big) = \sum_{\alpha\in\mathbb{Z}^n\cap[-N,N]} c_\alpha \mathrm{e}^{2\pi\mathrm{i}\alpha\cdot y},$$

where $N = (N_1,\ldots,N_n)$, $[-N,N] = \prod_{j=1}^n[-N_j,N_j]$, $c_\alpha = \prod_{j=1}^n c_{j,\alpha_j}$ for $\alpha = (\alpha_1,\ldots,\alpha_n)$, and $\alpha\cdot y = \sum_{j=1}^n \alpha_j y_j$. Thus,

$$P\circ\psi(x,t) = \sum_{\alpha\in\mathbb{Z}^n\cap[-N,N]} c_\alpha \mathrm{e}^{2\pi\mathrm{i}\alpha\cdot\psi(x,t)}.$$

Observe that

$$\frac{1}{T}\int_0^T c_0\,\mathrm{d}t - \int_{[0,1]^n} c_0\,ds = c_0 - c_0 = 0 \qquad \text{and that} \qquad \int_{[0,1]^n} c_\alpha \mathrm{e}^{2\pi\mathrm{i}\alpha\cdot s}\,\mathrm{d}s = 0$$

for all nonzero $\alpha \in \mathbb{Z}^n \cap [-N,N]$. Therefore, the term (**) equals

$$\Big|\sum_{\alpha\in(\mathbb{Z}^n\setminus\{0\})\cap[-N,N]} \frac{c_\alpha}{T}\int_0^T \mathrm{e}^{2\pi\mathrm{i}\alpha\cdot\psi(x,t)}\,\mathrm{d}t\Big|.$$

Using the notation J_k from Remark 8.8, for each nonzero $\alpha \in \mathbb{Z}^n \cap [-N,N]$ let

$$\phi_{\alpha,k}(x) = \sum_{j\in J_k}\alpha_j g_j(x) \quad \text{for each } k \in \{1,\ldots,\gamma\},$$

$$d(\alpha) = \max\big\{k \in \{1,\ldots,\gamma\} : \alpha_j \neq 0 \text{ for some } j \in J_k\big\},$$

and observe that

$$\alpha\cdot\psi(x,t) = \sum_{k=1}^{d(\alpha)} \phi_{\alpha,k}(x)t^k.$$

The set X is compact, the functions $\phi_{\alpha,1},\dots,\phi_{\alpha,d(\alpha)}$ are continuous on X, and by Remark 8.8, $\phi_{\alpha,d(\alpha)}$ has no zero in X because $t\mapsto\psi_1(x,t),\dots,t\mapsto\psi_n(x,t)$ are linearly independent over $\mathbb{Q}$ for each $x\in X$. Therefore, $|\phi_{\alpha,1}|,\dots,|\phi_{\alpha,d(\alpha)}|$ are bounded above, and $|\phi_{\alpha,d(\alpha)}|$ is bounded below by a positive constant. We may apply Lemma 8.6 to fix $T_0>0$ such that, for all $x\in X$ and $T\geq T_0$, the term (**) is bounded above by $\frac{\varepsilon}{4}$. $\square$

Let us fix the notation in view of Lemma 8.9. For this consider a cell

$$A=\{(x,t):x\in\Pi_m(A),t>a(x)\},$$

where $\Pi_m(A)$ is connected and open in $\mathbb{R}^m$. Define $f:A\to\mathbb{C}$ by

$$f(x,t)=\sum_{j\in J}f_j(x)\mathrm{e}^{\mathrm{i}\phi_j(x,t)},$$

where J is a nonempty finite index set, $(f_j)_{j\in J}$ is a family of analytic functions in $\mathcal{C}^{\exp}(\Pi_m(A))$, $(\phi_j)_{j\in J}$ is a family of distinct functions on $\Pi_m(A)\times\mathbb{R}$ that are polynomials in t with analytic coefficients in $\mathcal{S}(\Pi_m(A))$, and $\phi_j(x,0)=0$ for all $j\in J$ and $x\in\Pi_m(A)$.

Lemma 8.9 below is the parametric analogue of the presentation of the function $f(t)$ in Remark 6.4 as a nonconstant Laurent polynomial in $\mathrm{e}^{2\pi\mathrm{i}\widetilde{p}_1},\dots,\mathrm{e}^{2\pi\mathrm{i}\widetilde{p}_\ell}$, where the polynomial map $(\widetilde{p}_1(t),\dots,\widetilde{p}_\ell(t))$ is c.u.d. mod 1. Here in the parametric case it is technically more convenient to present $f(x,t)$ as a nonconstant Laurent polynomial (with coefficient functions in the parameter x) in $\mathrm{e}^{2\pi\mathrm{i}\psi_1(x,t)},\dots,\mathrm{e}^{2\pi\mathrm{i}\psi_n(x,t)}$, where the map $(\psi_1(x,t),\dots,\psi_n(x,t))$ is a monomial map in t that is c.u.d. mod 1 on certain compact sets of $\Pi_m(A)$.

LEMMA 8.9

With the notation just fixed above, we may express f as a composition

$$f(x,t)=F\big(x,\psi(x,t)\big)$$

on A, where for some $n\in\mathbb{N}$, $\psi=(\psi_1,\dots,\psi_n)$ is a monomial map in t with analytic coefficient functions in $\mathcal{S}(\Pi_m(A))$ and $F(x,z_1,\dots,z_n)$ is a Laurent polynomial in the variables $\mathrm{e}^{2\pi\mathrm{i}z_1},\dots,\mathrm{e}^{2\pi\mathrm{i}z_n}$ with coefficients $f_j(x)$, $j\in J$. If J is a singleton $\{j_0\}$ and if $\phi_{j_0}=0$, then $n=0$ and $F(x)=f_{j_0}(x)$. Otherwise we have $n>0$ and

(1) *there exists a set $B\subseteq\Pi_m(A)$ such that $\mathrm{vol}_m(\Pi_m(A)\setminus B)=0$ and, for any $x\in B$, $z\mapsto F(x,z)$ is nonconstant,*

(2) *for any open set $\Omega\subseteq\Pi_m(A)$ and any real number $\lambda<\mathrm{vol}_m(\Omega)$, there exists a real number T_0 and a compact set $K\subseteq\Omega\cap B$ such that $K\times[T_0,+\infty)\subseteq A$, $\mathrm{vol}_m(K)\geq\lambda$, and $\psi\restriction K\times[T_0,+\infty)$ is c.u.d. mod 1 on K.*

Proof
Since the functions ϕ_j, $j \in J$, are distinct, it is only possible to have $\phi_j = 0$ for all $j \in J$ when J is a singleton $\{j_0\}$, and in this case we have $f(x,t) = f_{j_0}(x)$. We may now assume that $\phi_j \neq 0$ for some $j \in J$. In this case, since $\phi_j(x,0) = 0$ for all $j \in J$ and all $x \in \Pi_m(A)$,

$$d := \max\{\deg \phi_j : j \in J\}$$

is a positive integer. For each $j \in J$, write

$$\phi_j(x,t) = \sum_{k=1}^{d} \phi_{j,k}(x) t^k$$

with $\phi_{j,k} \in \mathcal{S}(\Pi_m(A))$. For each $k \in \{1,\ldots,d\}$, fix $\Gamma_k \subseteq J$ such that $(\phi_{\gamma,k})_{\gamma\in\Gamma_k}$ is a basis over $\mathbb{Q}$ of the span over $\mathbb{Q}$ of the family $(\phi_{j,k})_{j\in J}$ (as functions of x), and let

$$\Gamma = \big\{(\gamma,k) : k \in \{1,\ldots,d\}, \gamma \in \Gamma_k\big\}.$$

We may fix a positive integer η such that, for each $(j,k) \in J \times \{1,\ldots,d\}$,

$$\phi_{j,k} = \sum_{\gamma\in\Gamma_k} \frac{\alpha_{j;\gamma,k}}{\eta} \phi_{\gamma,k}$$

for a unique tuple of integers $(\alpha_{j;\gamma,k})_{\gamma\in\Gamma_k}$. With this notation we have

$$\begin{aligned} f(x,t) &= \sum_{j\in J} f_j(x) \mathrm{e}^{\mathrm{i}\sum_{k=1}^{d} \phi_{j,k}(x)t^k} = \sum_{j\in J} f_j(x) \mathrm{e}^{\mathrm{i}\sum_{k=1}^{d}\sum_{\gamma\in\Gamma_k} \frac{\alpha_{j;\gamma,k}}{\eta}\phi_{\gamma,k}(x)t^k} \\ &= \sum_{j\in J} f_j(x) \prod_{(\gamma,k)\in\Gamma} (\mathrm{e}^{2\pi\mathrm{i}\psi_{\gamma,k}(x)})^{\alpha_{j;\gamma,k}} = F\big(x, \big(\psi_{\gamma,k}(x)\big)_{(\gamma,k)\in\Gamma}\big), \end{aligned}$$

where for each $(\gamma,k) \in \Gamma$, $\psi_{\gamma,k}(x,t) = \frac{\phi_{\gamma,k}(x)t^k}{2\pi\eta}$ and

$$F\big(x, (z_{\gamma,k})_{(\gamma,k)\in\Gamma}\big) = \sum_{j\in J} f_j(x) \prod_{(\gamma,k)\in\Gamma} (\mathrm{e}^{2\pi\mathrm{i} z_{\gamma,k}})^{\alpha_{j;\gamma,k}}.$$

For each $j \in J$, f_j is a nonzero analytic function on the connected and open set $\Pi_m(A)$, so the set

$$U := \big\{x \in \Pi_m(A) : f_j(x) \neq 0 \text{ for all } j \in J\big\}$$

satisfies $\mathrm{vol}(\Pi_m(A) \setminus U) = 0$. The fact that ϕ_j, $j \in J$, are distinct functions implies that $((\alpha_{j;\gamma,k})_{(\gamma,k)\in\Gamma})_{j\in J}$ is a family of distinct tuples in $\mathbb{Z}^\Gamma$. As a consequence, for each $x \in U$ the trigonometric polynomial $z \mapsto F(x,z)$ is nonconstant.

Observe that, since $(\phi_{\gamma,k})_{\gamma\in\Gamma_k}$ is independent over $\mathbb{Q}$ (as functions of x), for each $k \in \{1,\dots,d\}$ and nonzero tuple $c = (c_\gamma) \in \mathbb{Z}^{\Gamma_k}$, $\sum_{\gamma\in\Gamma_k} c_\gamma \phi_{\gamma,k}$ is a nonzero analytic function on $\Pi_m(A)$, so the set $\{x \in U : \sum_{\gamma\in\Gamma_k} c_\gamma \psi_{\gamma,k}(x) = 0\}$ cannot have a positive measure, and the set

$$B := U \setminus \Big(\bigcup_{k=1}^{d} \bigcup_{c\in\mathbb{Z}^{\Gamma_k}\setminus\{0\}} \Big\{x \in U : \sum_{\gamma\in\Gamma_k} c_\gamma \phi_{\gamma,k}(x) = 0\Big\}\Big)$$

satisfies $\mathrm{vol}_m(\Pi_m(A) \setminus B) = 0$ as well. This gives (1), since $B \subset U$.

On the other hand, from the definition of B we see that, for each $k \in \{1,\dots,d\}$ and for each $x \in B$, the family of numbers $(\phi_{\gamma,k}(x))_{(\gamma,k)\in\Gamma}$ is linearly independent over $\mathbb{Q}$, and by Remark 8.8, for each $x \in B$ the family of functions $(t \mapsto \psi_{\gamma,k}(x,t))_{(\gamma,k)\in\Gamma}$ is linearly independent over $\mathbb{Q}$. Given an open set $\Omega \subseteq \Pi_m(A)$ and any positive real number λ with $\lambda < \mathrm{vol}_m(\Omega) = \mathrm{vol}_m(\Omega \cap B)$, the inner regularity of the Lebesgue measure shows that we may fix a compact set $K \subseteq \Omega \cap B$ with $\mathrm{vol}_m(K) \ge \lambda$. Since K is compact and $a(x)$ is continuous, we may fix T_0 sufficiently large so that $K \times [T_0,+\infty) \subseteq A$. Proposition 8.7 then shows that the restriction of $\psi := (\psi_{\gamma,k})_{(\gamma,k)\in\Gamma}$ to $K \times [T_0,+\infty)$ is c.u.d. mod 1 on K, which completes the proof of (2). □

The following Lemma 8.10 is the parametric version of Proposition 6.5(1). It will be used in the proof of Proposition 8.2.

LEMMA 8.10

Consider $f(x,t) = F(x,\psi(x,t))$ as given in Lemma 8.9, with $z \mapsto F(x,z)$ nonconstant for some $x \in \Pi_m(A)$. Then there exist $\varepsilon > 0$, $\delta > 0$, a strictly increasing sequence $(t_j)_{j\in\mathbb{N}}$ in $\mathbb{R}$ diverging to $+\infty$, a compact set $K \subset X$, and a sequence $(X_j)_{j\in\mathbb{N}}$ of Lebesgue measurable subsets of $K \subseteq \Pi_m(A)$, with, for any $j \in \mathbb{N}$, $\mathrm{vol}_m(X_j) \ge \delta$, $X_{2j+1} \subseteq X_{2j}$, such that, for all $j \in \mathbb{N}$, for all $x_0 \in X_{2j}$ and $x_1 \in X_{2j+1}$,

$$\big|f(x_0,t_{2j})\big| \ge \varepsilon \qquad \text{and} \qquad \big|f(x_0,t_{2j}) - f(x_1,t_{2j+1})\big| \ge \varepsilon.$$

Proof

Since $z \mapsto F(x,z)$ is nonconstant and 1-periodic in each of the components of $z = (z_1,\dots,z_n)$, one may find $v_0, v_1 \in [0,1)^n$ such that $f(x,v_0) \ne f(x,v_1)$, and thus, assuming for instance that $f(x,v_0) \ne 0$, one may fix $\varepsilon > 0$, an open subset U of $\Pi_m(A)$ containing x, and boxes $I_0, I_1 \subseteq [0,1)^n$, respectively, containing v_0, v_1 such that $\mathrm{dist}(F(U \times I_0), 0) \ge \varepsilon$ and $\mathrm{dist}(F(U \times I_0), F(U \times I_1)) \ge \varepsilon$.

By Lemma 8.9(2) we may fix a compact set $K \subseteq U$ and $T_0 \in \mathbb{R}$ such that $\mathrm{vol}_m(K) > 0$, $K \times [T_0,+\infty) \subseteq A$, and $\psi \restriction K \times [T_0,+\infty)$ is c.u.d. mod 1 on K.

Define

$$\delta = \frac{1}{2}\operatorname{vol}_m(K)\operatorname{vol}_n(I_0)\min\Big\{1, \frac{1}{2}\operatorname{vol}_n(I_1)\Big\}.$$

Remark 8.5 shows that we may fix $k_0 \in \mathbb{N}$, $2^{k_0} > T_0$, such that, for all $k \geq k_0$, all $i \in \{0, 1\}$, and all $x \in K$,

$$\operatorname{vol}_1\Big(\big\{t \in [2^k, 2^{k+1}) : \big\{\psi(x,t)\big\} \in I_i\big\}\Big) \geq 2^{k-1}\operatorname{vol}_n(I_i).$$

Let us now construct t_0, t_1 and the corresponding sets $X_1 \subset X_0 \subset K$. For this we consider

$$E_0 := \big\{(x,t) \in K \times [2^{k_0}, 2^{k_0+1}) : \big\{\psi(x,t)\big\} \in I_0\big\}.$$

By integrating first in the variable t and then in the variable x, Fubini's theorem gives

$$\operatorname{vol}_{m+1}(E_0) = \int_{x\in K}\operatorname{vol}_1\Big(\big\{t : (x,t) \in E_0\big\}\Big)\,\mathrm{d}x \geq \operatorname{vol}_m(K)2^{k_0-1}\operatorname{vol}_n(I_0).$$

But, by integrating first in the variable x and then in the variable t, Fubini's theorem also gives

$$\operatorname{vol}_{m+1}(E_0) = \int_{2^{k_0}}^{2^{k_0+1}}\operatorname{vol}_m\Big(\big\{x : (x,t) \in E_0\big\}\Big)\,\mathrm{d}t.$$

It follows that we may certainly choose $t_0 \in [2^{k_0}, 2^{k_0+1})$ to define

$$X_0 = \big\{x \in K : (x, t_0) \in E_0\big\}$$

so that

$$\operatorname{vol}_m(X_0) \geq \frac{\operatorname{vol}_{m+1}(E_0)}{2^{k_0}} \geq \frac{1}{2}\operatorname{vol}_m(K)\operatorname{vol}_n(I_0) \geq \delta. \tag{61}$$

Now denote $k_0 + 1$ by k_1. Then $2^{k_1} > t_0$. We apply the same construction as above but with I_1 instead of I_0, $[2^{k_1}, 2^{k_1+1})$ instead of $[2^{k_0}, 2^{k_0+1})$, and X_0 instead of K. For this we define

$$E_1 = \big\{(x,t) \in X_0 \times [2^{k_1}, 2^{k_1+1}) : \big\{\psi(x,t)\big\} \in I_1\big\},$$

and then we choose, with the same argument as above using Fubini's theorem on E_1, some $t_1 \in [2^{k_1}, 2^{k_1+1})$ and define

$$X_1 = \big\{x \in X_0 : (x, t_1) \in E_1\big\}$$

so that, in conjunction with (61),

$$\mathrm{vol}_m(X_1) \geq \frac{1}{2}\,\mathrm{vol}_m(X_0)\,\mathrm{vol}_n(I_1) \geq \frac{1}{4}\,\mathrm{vol}_m(K)\,\mathrm{vol}_n(I_0)\,\mathrm{vol}_n(I_1) \geq \delta.$$

For $j \geq 1$, the pairs (t_{2j}, t_{2j+1}) and (X_{2j}, X_{2j+1}) are defined in the same way, t_{2j} being constructed from t_{2j-1}, X_{2j} from K, t_{2j+1} from t_{2j}, and X_{2j+1} from X_{2j}. $\square$

We can finally prove Proposition 8.2.

Proof of Proposition 8.2

Let $p \in [1, +\infty]$ and $f \in \mathcal{C}^{\exp}(X \times \mathbb{R})$ for a subanalytic set $X \subseteq \mathbb{R}^m$, and suppose that $(f_y)_{y\in\mathbb{R}}$ is Cauchy in $L^p(X)$ as $y \to +\infty$. Since $L^p(X)$ is complete, there exists a function $h \in L^p(X)$ such that

$$\lim_{y\to+\infty} \|f_y - h\|_p = 0, \tag{62}$$

and there exists a sequence $(y_j)_{j\in\mathbb{N}}$ in $\mathbb{R}$ tending to $+\infty$ such that

$$\lim_{j\to+\infty} f(x, y_j) = h(x) \quad \text{for almost all } x \in X. \tag{63}$$

(See for instance Rudin [26, Theorems 3.11 and 3.12].)

Apply Theorem 5.2 to $f(x, y)$ with respect to y. Let $\mathcal{A}$ be the collection of cells A given by the preparation that are open in $\mathbb{R}^{m+1}$ and of the form

$$A = \big\{(x, y) : x \in \Pi_m(A), y > a(x)\big\},$$

and put $X_0 = \bigcup\{\Pi_m(A) : A \in \mathcal{A}\}$. Since $\mathrm{vol}_m(X \setminus X_0) = 0$, it suffices to focus on one $A \in \mathcal{A}$ and prove that the conclusion of the theorem holds for $f \restriction A$. Write f as a finite sum

$$f(x, y) = \sum_{j\in J} T_j(x, y)$$

on A with each term of the form $T_j(x, y) = y^{r_j}(\log y)^{s_j} g_j(x, y)$ specified in Remark 5.3; thus, $g_j \in \mathcal{C}^{\exp}(A)$ with $|g_j(x, y)| \leq \eta_j(x)$ on A for some continuous function $\eta_j : \Pi_m(A) \to [0, +\infty)$, and $g_j(x, y) = f_j(x)\mathrm{e}^{\mathrm{i}\phi_j(x,y)}$ when $r_j \geq -1$, with $\phi_j(x, y)$ distinct polynomials in $y^{1/d}$ for some integer $d \geq 0$ such that $\phi_j(x, 0) = 0$ for all $x \in \Pi_m(A)$. Each function f_j can be taken to be analytic on A by Remark 5.8 and not identically zero, and $\Pi_m(A)$ is connected since A is a subanalytic cell. We claim that there exists $g \in \mathcal{C}^{\exp}(\Pi_m(A))$ such that $\lim_{y\to+\infty} f(x, y) = g(x)$ for all $x \in \Pi_m(A)$. This claim and (63) imply that $g(x) = h(x)$ for almost all $x \in \Pi_m(A)$, and hence $\lim_{y\to+\infty} \|f_y \restriction \Pi_m(A) - g\|_p = 0$ by (62). So we will be done once we prove the claim.

Let $E = \{(r_j, s_j) : j \in J\}$, and for each $(r,s) \in E$ let $J_{(r,s)} = \{j \in J : (r_j, s_j) = (r,s)\}$. Thus,

$$f(x,y) = \sum_{(r,s)\in E} y^r (\log y)^s S_{(r,s)}(x,y), \tag{64}$$

where for each $(r,s) \in E$,

$$S_{(r,s)}(x,y) = \sum_{j\in J_{(r,s)}} g_j(x,y). \tag{65}$$

For each $(r,s) \in E$, define $\eta_{(r,s)} : \Pi_m(A) \to [0,+\infty)$ by $\eta_{(r,s)}(x) = \sum_{j\in J_{(r,s)}} \eta_j(x)$, and observe that $\eta_{(r,s)}$ is continuous and that $|S_{(r,s)}(x,y)| \le \eta_{(r,s)}(x)$ on A.

Let $(\overline{r},\overline{s})$ be the lexicographic maximum element of E. If $\overline{r} < 0$, then $\lim_{y\to+\infty} f(x,y) = 0$ for all $x \in \Pi_m(A)$, and we are done. If $\overline{r} = \overline{s} = 0$ and J is a singleton, say, $J = \{j_0\}$, and $\phi_{j_0} = 0$, then

$$f(x,y) = f_{j_0}(x) + \sum_{(r,s)\in E\setminus\{(0,0)\}} y^r (\log y)^s g_j(x,y),$$

with $r < 0$ for all $(r,s) \in E \setminus \{(0,0)\}$, so $\lim_{y\to+\infty} f(x,y) = f_{j_0}(x)$ on $\Pi_m(A)$, and we are also done. The two remaining cases are when $\overline{r} > 0$ or $\overline{s} > 0$, or when $\overline{r} = \overline{s} = 0$ and $\phi_j \ne 0$ for some $j \in J_{(0,0)}$. We will complete the proof by showing that these two remaining cases are impossible.

We may assume that $\overline{r} \ge 0$, since this is a common assumption of the two remaining cases. Note that

$$S_{(\overline{r},\overline{s})}(x,y^d) = \sum_{j\in J_{(\overline{r},\overline{s})}} f_j(x) \mathrm{e}^{\mathrm{i}\phi_j(x,y^d)}$$

is of the form hypothesized in Lemma 8.9. Therefore, we can apply Lemma 8.10 and find $\varepsilon > 0$, $\delta > 0$, a compact set $K \subset \Pi_m(A)$, a strictly increasing sequence $(y_j)_{j\in\mathbb{N}}$ in $(1,+\infty)$ tending to $+\infty$ with $K \times [y_0,+\infty) \subset A$, and a sequence $(X_j)_{j\in\mathbb{N}}$ of Lebesgue measurable subsets of K such that, for all $j \in \mathbb{N}$, $\mathrm{vol}_m(X_j) \ge \delta$, $X_{2j+1} \subset X_{2j} \subset K$, and

$$\forall x \in X_{2j+1}, \quad \left|S_{(r,s)}(x,y_{2j})\right| \ge 2\varepsilon,$$
$$\left|S_{(r,s)}(x,y_{2j+1}) - S_{(r,s)}(x,y_{2j})\right| \ge 3\varepsilon.$$

The set K is compact, each function $\eta_{(r,s)}$ is continuous, and it holds that $\lim_{y\to+\infty} y^{r-\overline{r}}(\log y)^{s-\overline{s}} = 0$ for all $(r,s) \in E \setminus \{(\overline{r},\overline{s})\}$, so by replacing $(y_j)_{j\in\mathbb{N}}$ with a tail of the sequence, we may assume that, for all $y \ge y_0$,

$$\max\Big\{ \sum_{(r,s)\in E\setminus\{(\overline{r},\overline{s})\}} y^{r-\overline{r}}(\log y)^{s-\overline{s}} \eta_{(r,s)}(x) : x \in K \Big\} \le \varepsilon.$$

Observe that, for all $j \in \mathbb{N}$ and $x \in X_{2j}$,

$$\Big|S_{(\overline{r},\overline{s})}(x, y_{2j}) + \sum_{(r,s)\in E\setminus\{(\overline{r},\overline{s})\}} y_{2j}^{r-\overline{r}}(\log y_{2j})^{s-\overline{s}} S_{(r,s)}(x, y_{2j})\Big| \geq 2\varepsilon - \varepsilon = \varepsilon,$$

so

$$\begin{aligned}\big|f(x, y_{2j})\big| &= y_{2j}^{\overline{r}}(\log y_{2j})^{\overline{s}}\Big|S_{(\overline{r},\overline{s})}(x, y_{2j}) \\ &\qquad + \sum_{(r,s)\in E\setminus\{(\overline{r},\overline{s})\}} y_{2j}^{r-\overline{r}}(\log y_{2j})^{s-\overline{s}} S_{(r,s)}(x, y_{2j})\Big| \\ &\geq y_{2j}^{\overline{r}}(\log y_{2j})^{\overline{s}}\varepsilon.\end{aligned}$$

In consequence, if $\overline{r} > 0$ or $\overline{s} > 0$, then

$$\|f_{y_{2j}} - h\|_p \geq \|f_{y_{2j}}\|_p - \|h\|_p \geq y_{2j}^{\overline{r}}(\log y_{2j})^{\overline{s}}\varepsilon\delta - \|h\|_p \underset{n\to+\infty}{\longrightarrow} +\infty,$$

which contradicts (62). So we may suppose that $\overline{r} = \overline{s} = 0$ and $\phi_j \neq 0$ for some $j \in J_{(0,0)}$. Thus, on A

$$f(x, y) = S_{(0,0)}(x, y) + \sum_{(r,s)\in E\setminus\{(0,0)\}} y^r(\log y)^s S_{(r,s)}(x, y).$$

It follows that, for all $j \in \mathbb{N}$ and $x \in X_{2j+1}$,

$$\begin{aligned}\big|f(x, y_{2j}) - f(x, y_{2j+1})\big| &\geq \big|S(x, y_{2j}) - S(x, y_{2j+1})\big| \\ &\qquad - \Big|\sum_{(r,s)\in E\setminus\{(0,0)\}} y_{2j}^r(\log y_{2j})^s S_{(r,s)}(x, y_{2j})\Big| \\ &\qquad - \Big|\sum_{(r,s)\in E\setminus\{(0,0)\}} y_{2j+1}^r(\log y_{2j+1})^s S_{(r,s)}(x, y_{2j+1})\Big| \\ &\geq 3\varepsilon - \varepsilon - \varepsilon = \varepsilon.\end{aligned}$$

Finally we obtain, for all $j \in \mathbb{N}$,

$$\|f_{2j} - f_{y_{2j+1}}\|_p \geq \varepsilon\delta,$$

which contradicts the fact that $(f_y)_{y\in\mathbb{R}}$ is Cauchy in $L^p(X)$ as $y \to +\infty$. □

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Cluckers

Laboratoire Painlevé, CNRS UMR 8524, Cité Scientifique, Université de Lille, Villeneuve d'Ascq, France; and Department of Mathematics, Katholieke Universiteit Leuven, Leuven, Belgium; Raf.Cluckers@math.univ-lille1.fr

Comte

Université Grenoble Alpes, Université Savoie Mont Blanc, CNRS, LAMA, Chambéry, France; Georges.Comte@univ-smb.fr

Miller

Department of Mathematics and Economics, Emporia State University, Emporia, Kansas, USA; dmille10@emporia.edu

Rolin

Institut de Mathématiques de Bourgogne, CNRS UMR 5584, Université de Bourgogne Franche-Comté, Dijon, France; jean-philippe.rolin@u-bourgogne.fr

Servi

Institut de Mathématiques de Jussieu, Unité de Formation et de Recherche de Mathématiques, Paris Rive Gauche Université Paris Diderot (Paris 7), Paris, France; tamara.servi@math.univ-paris-diderot.fr